Choices and Strategies

Knowing how to play the game

Everything is mathematical

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Jordi Deulofeu

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*There is no branch of mathematics,
however abstract, which may not some day
be applied to phenomena of the real world.*

N. Lobachevsky

*If people do not believe that mathematics is simple, it is
only because they do not realise how complicated life is.*

John von Neumann

Preface

What is the connection between games and mathematics? Are mathematical games just for entertainment purposes or can they also be used to model real life situations? When a game is analysed from a mathematical perspective, what information is required and what can be learnt? Can mathematics be used to analyse aspects of human behaviour and help with making decisions?

These are just some of the questions this book attempts to answer. This is a book about mathematics and games, which in contrast to others dealing with the same topics, is not made up of a collection of various games requiring a range of skills, but instead is based around the collection of mathematical concepts, processes and theories that can be developed, based on the analysis of certain games.

The approach to the material in the book tries to show that dichotomies such as serious or recreational mathematics, and pure or applied mathematics, can in fact be two sides of the same coin – or even better, the four sides of a tetrahedron. The mathematical study of games, something which initially appears to be entertainment and the analysis of which results in mathematics for pure intellectual pleasure, can become one of the most relevant branches of mathematics to real-world situations thanks to game theory.

After a first chapter covering the history of the subject in order to show the historical relationship between mathematics and games, the next two chapters deal with games that firstly do not involve chance (so-called complete information games) and then move on to games of chance. Chapter 2 has some examples of small-scale strategy games and shows how a game can be analysed to determine a way of playing that always makes it possible to win (a winning strategy) and investigates the mathematics involved in this analysis. Chapter 3 discusses the basic mathematics of chance, based on betting games that require the calculation of how likely events are, a process that forms the basis of probability theory.

The final two chapters provide an introduction to game theory, the branch of mathematics founded by John von Neumann in the early part of the twentieth century. The theory studies aspects of human behaviour in order to try to optimise decision making in fields as diverse as economics, politics, military organisations and animal behaviour. The theory makes use of games as mathematical models that simulate real situations.

Game theory analyses certain dilemmas, such as the game of *chicken* – up to what point should a risk be taken in order to win? – and the prisoner's dilemma

– keep quiet or inform? Both these classic conundrums reflect the circumstances present in many events in our world, where the tension between confrontation and cooperation makes it hard to make the best decisions. Even if mathematics does not provide definitive solutions to these dilemmas, it shows, by means of quantifying the various possibilities, the risks of confrontation and the advantages of cooperation.

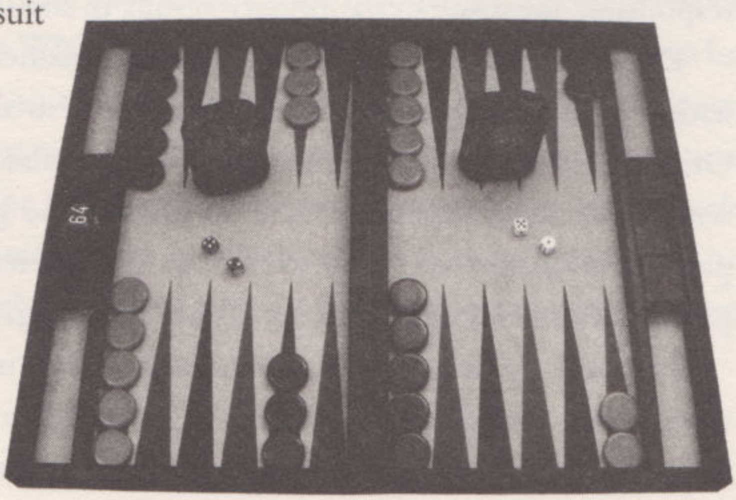
Chapter 1

A Brief History of the Relationship Between Mathematics and Games

Life is worth living to play
the finest games … and win.
Plato

Is mathematics always serious or can it be something playful? Is pure maths the only true discipline or is applied maths its equal? Of course the answers for both these questions can be yes and no. However this should rightly be interpreted as attempting to avoid the question, so instead we will attempt throw more light on the subject by explaining the reason for asking them in the first place.

The debate as to whether mathematics exists for its own ends in an attempt to solve its own problems or whether it arises from problems that come up in other disciplines or areas is age old. Looking back at the history of the science can help shed some light on the issue. The mathematics of ancient Egypt and Babylon was essentially an applied science, as is shown in the records that have been passed down. However, with the Greeks, something changed. Mathematics became a tool for showing absolute truths – a pure science that deals with abstract entities, such as numbers and shapes, although applications of these often crop up unexpectedly on countless occasions, both in everyday life and in the pursuit of other sciences.



The playful nature of games does not exclude them from involving calculations; on the contrary it is often the player who makes the best calculations who wins the game.

It can be argued that mathematics, in the broadest sense of the word, exists as an attempt to solve problems and answer questions about our world. However, since mathematics is a uniquely human activity, it is also dependent on the culture in which its practitioners live and work, and it is this culture that determines which problems are significant enough to need solutions.

Serious mathematics and playful mathematics, pure and applied

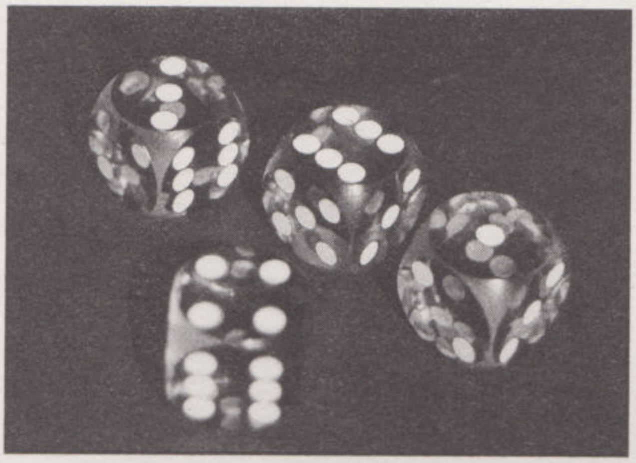
In his lecture *The Role of Mathematics in Science and Society*, John von Neumann, one of the most important figures in this book, explained how many great mathematical ideas have been developed without thinking about any applications or considering how they could be useful. Instead the theories, models and methods developed by mathematicians have shown themselves to be useful for solving problems or answering questions from the most diverse areas of knowledge. At the same time, many mathematical ideas have become all pervading; although mathematics seems far removed from reality, it is found in almost all spheres of life.

In no way can von Neumann be described as a mathematician who did not value finding applications for his theories, after all it is not for nothing that he is one of the inventors of game theory, a branch of applied mathematics. He explained how many triumphs of science have occurred when researchers have stopped investigating what can be useful and have let themselves be guided by curiosity in search of intellectual elegance. In fact, at the end of his lecture, von Neumann noted that scientific progress had gone beyond what humanity would have achieved had it strictly limited research to the useful, and that this *laissez faire* approach has been responsible for some extraordinary results in the field of mathematics.

Drawing a parallel with the usefulness of mathematics, let us turn to the playful nature of this discipline. Can a science that is so abstract also be a source of fun? Once again, the history of mathematics helps answer the question. In this chapter we will see how the mathematics of puzzles and games are seen in practically all periods of history, and have even been the inspiration for new theories, such as probability theory, graph theory and of course, game theory.

Puzzles, games and mathematical problems have one thing in common – they represent an intellectual challenge, a dare the acceptance of which leads the player to make great efforts to solve the problem and beat their

opponent. This striving looks hard work and and even boring to the untutored spectator. However for those who enjoy intellectual challenges and games in which it is necessary to 'think', such activity provides a strong source of satisfaction. This is because, as Miguel de Guzmán explains, mathematics is always a game, although it is also many other things at the same time.



Many conventional games can be analysed from the perspective of game theory.

Just because mathematics is an important pursuit in its own right, has a multiple of applications in diverse areas of life, and is frequently difficult (as is playing some of the best games), it does not necessarily mean that mathematics is boring. It is true that certain teaching practices can lead us to think this. Carrying out meaningless exercises has little to do with mathematics. However anyone who has managed to get into mathematics knows that it can be exciting and a lot of fun.

A brief overview of the history of mathematics and games will show that the playful elements have always been present throughout time, from ancient Egypt through to the 21st century. Although the word *game* is often used to refer to any fun individual or collective activity, from now on it will be used to distinguish between mathematical puzzles and games. While puzzles are problems of a playful nature to be solved by one person, a game is an activity in which at least two people participate, with the players' main goal being to beat their opponents. Secondly, when we come to analyse games, our goal will be to determine the winning strategies, where such strategies exist (in the case of finite games in which chance does not intervene) or strategies that increase the probability of winning (in the case of games where there is an element of chance).

Mathematics and games up to the 17th century

Since its origins, the history of mathematics is full of references to games. In fact, ever since humans began to play games and, in parallel to this, to develop mathematics, it was impossible to separate what we call serious mathematics from playful mathematics or mathematical puzzles prior to the 17th century, so intertwined were both activities. In 1612 the first book exclusively dedicated to mathematical puzzles was published in France: *Problèmes Plaisants et Délectables qui se Font par les Nombres* by Claude-Gaspar Bachet de Méziriac. From that point onwards, the two aspects of mathematics were to gradually separate. Nevertheless the two would meet frequently, such as in Fermat and Pascal's pioneering work on probability and an interest in puzzles shared by many great mathematical figures—from Newton to Euler to Gauss. Finally, the very serious maths of game theory was formulated in the middle of the 20th century.

Games and mathematics in the ancient civilisations

Board games and recreational problems were already around in the two great ancient civilisations of Babylon and Egypt, for whom mathematics was essentially of a practical nature. In terms of the former, Egyptian Senet and the Royal Game of Ur in Babylon are the two first records of board games, which have been passed down to us. We can find evidence in one of the oldest known documents of Egyptian mathematics, the *Rhind Papyrus*, dated to around 1650 B.C. and discovered in the great tomb of Ramesses II around 1850 A.D. In 1856 it was acquired in Luxor by Alexander Henry Rhind and is currently conserved at the British Museum in London. Alongside practical calculation problems to do with distribution and measurement, the document also contains mathematical problems without such a context and which hint at recreational aspects.

As an example, the 24th problem of the papyrus states: *Aha, a quantity plus one seventh of it makes 19*, a statement whose modern translation would be: find a number that gives the number 19 when added to one seventh of its value. A written solution to this problem – which could be solved easily using linear equations, although that method was evidently unknown to the Egyptians – is provided by Ahmes, the author of the papyrus, using an interesting technique called the method of false position. This was used by the ancients to solve many arithmetical problems. In the case in question, it is applied as follows: Ahmes imagines that 7 is the solution and makes the following calculation: $7 + 7 \cdot 1/7 = 8$.

A mural in the antechamber of the tomb of Queen Nefertari, wife of Ramesses II, show her playing a game of Senet.

The result is not 19 so he tries to discover by how much the number 8 must be multiplied to give 19, that is to say he divides 19 by 8, which in Egyptian mathematics is done as follows:

(8 x) 2 ---------- 16
(8 x) 1/4 --------- 2
(8 x) 1/8 --------- 1.

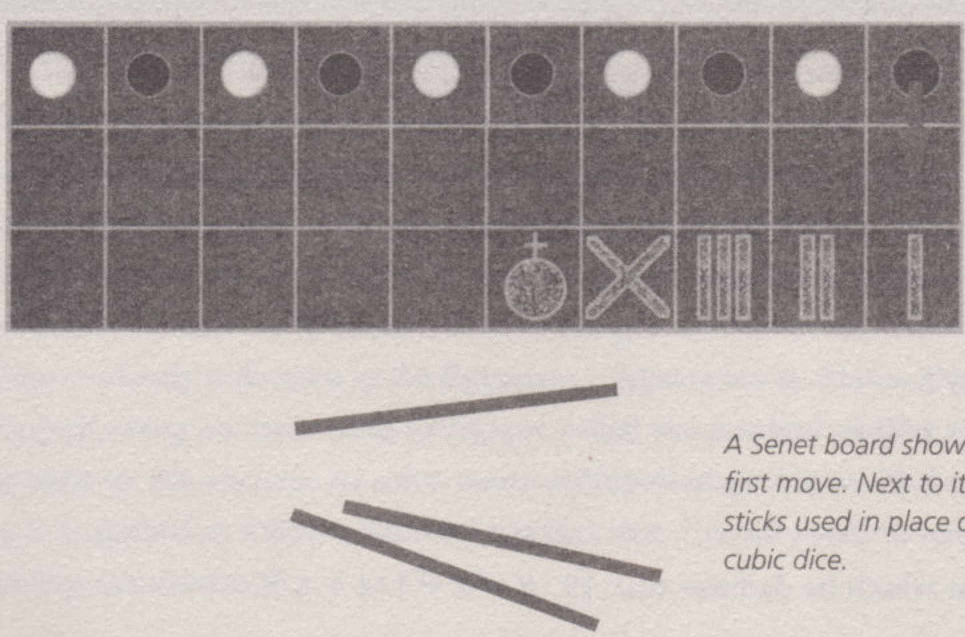
From which he deduces that: 19 : 8 = 2 + 1/4 + 1/8.

He then multiplies 7 by: 2 + 1/4 + 1/8, and gets: 14 + (1 + 1/2 + 1/4) + (1/2 + 1/4 + 1/8) = 16 + 1/2 + 1/8, or rather 16 + 5/8, or rather, 16.625.

The reader can see two features of Egyptian mathematics: the use of operations and the use of fractions. To carry out the division, the scribe Ahmes finds three powers of two which make 19 (16, 2 and 1) and he takes the eighth part of each, 2, 1/4, 1/8, and adds these values.

SENET, AN AGE OLD GAME

Senet is one of the oldest known board games. There are records that show it was played in ancient Egypt. This evidence takes the form of a variety of archaeological remains found in the tombs of both royalty and the public, which includes pictures and mosaics depicting people playing the game. However, its precise rules are unknown, although a reconstruction was made in 1991 by T Kendall and R May. The pair also observed that Senet was of great importance in funeral rights, with the deceased being required to play a game against their destiny in the presence of the god Osiris. In the *Book of the Dead* it is suggested that the afterlife of the deceased depends on the result of this game. The game consists of each player racing to remove their opponents' 7 counters from the board. Four small sticks, flat on one side and convex on the other, are used instead of dice. They are thrown simultaneously, giving five possible results depending on the number of sticks landing flat side up.

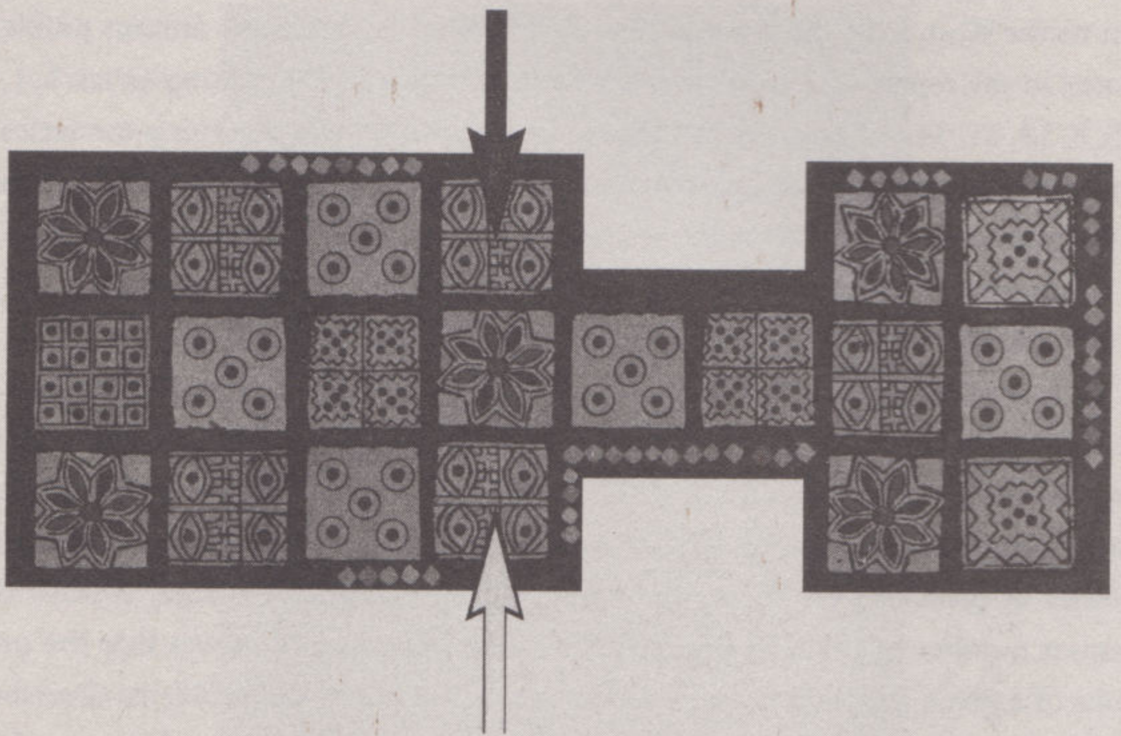


A Senet board showing the first move. Next to it are the sticks used in place of a cubic dice.

THE ROYAL GAME OF UR, OVER 4,000 YEARS OF HISTORY

Together with the Egyptian game of Senet, this is one of the oldest known board games. An ornate board was discovered in the Sumerian city of Ur by the British architect Sir Leonard Woolley around 1920. It is over 4,000 years old and is currently held in the British Museum in London. It is assumed that the game was played by royals and the nobility, and the fact that it was found in tombs suggests that it may have accompanied the deceased to allow them to play in the afterlife.

Like Senet, its rules are also unknown, although from the remains that have been found – as well as the board, there was a collection of 7 white mother-of-pearl and 7 black slate counters, and 6 dice in the shape of a triangular pyramid) it is assumed it was a racing game. The curious shape of the board – 20 boxes made up of a 3 x 2 and a 3 x 4 rectangle, joined by another 1 x 2 rectangle – provides strong hints at the path taken by the counters.



Board of the Royal Game of Ur showing the probable first move of each player.

For calculations with fractions, the scribe has only used unit fractions, also known as Egyptian fractions, which have the unit numerator. This curious arithmetic using fractions devised by the Egyptians has been studied by a number of famous mathematicians at various points in time, including the Italian Leonardo de Pisa, better known as Fibonacci (1175–1250), one of the great mathematicians of the

Middle Ages and the first to show the viability of the Egyptian method; the Englishman James Joseph Sylvester (1814–1897), who discovered new methods for expressing a fraction based on the sum of unit fractions; and the Hungarian Paul Erdös (1913–1996), one of the most prolific mathematicians of the 20th century, who was especially interested in number theory and who devised a great number of open problems related to Egyptian fractions, for which he provided some solutions.

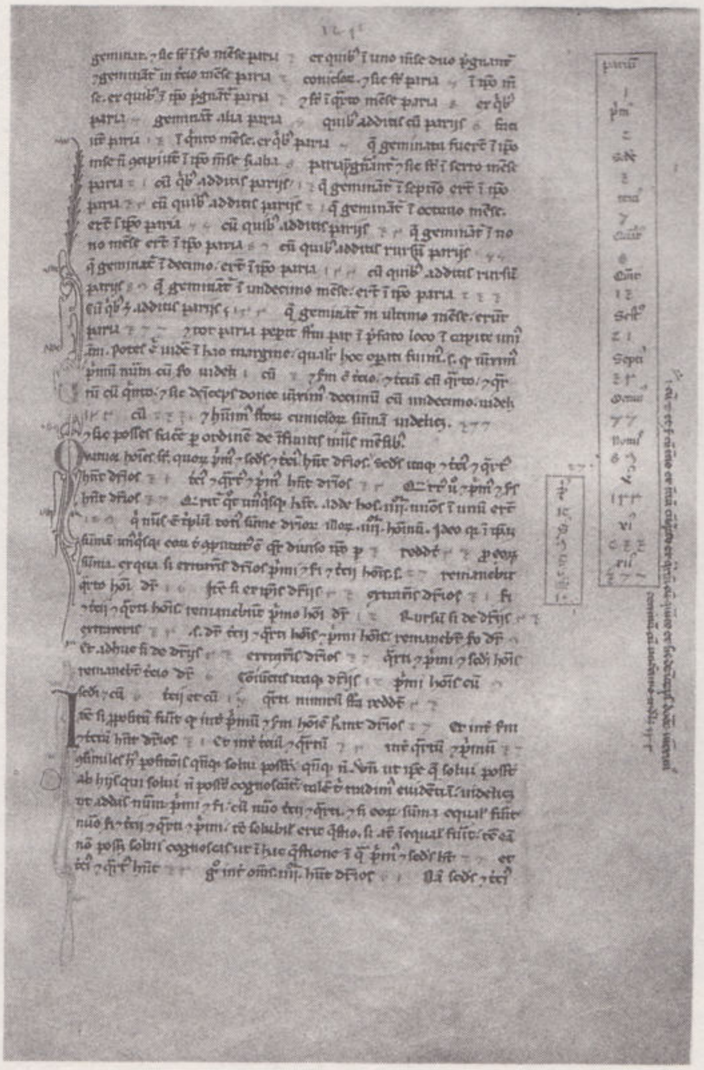
Games and mathematics in the middle ages

In this brief overview of the relationship between mathematics and games, it is only possible to highlight some of the most interesting moments from its history. Let us now make the leap into the 13th century and consider a number of highlights from this period. Leonardo de Pisa, better known as Fibonacci (1175–1250) and author of *Liber Abaci* (1202), a work that introduced the decimal position numbering system to the West, lived during this century. His book includes the famous problem related to the reproduction of rabbits, which generates the interesting series: 1, 1, 2, 3, 5, 8, 13, 21, 34... known as the Fibonacci series. The rule governing the series is extremely simple: after the first two terms, which are both 1, each term is the sum of the previous two. However, the series possess some fascinating properties, such as its connection with the golden ratio $(\Phi = (1+\sqrt{5})/2)$, the limit of the series a_n / a_{n-1}, where n tends to infinity, where a_n is a term for the Fibonacci series.

In one of his key works, *Liber Quadratorum* (*The Book of Squares*), published in 1225, Fibonacci comments on the mathematics tournament that took place in the court of Frederick II of Sicily, in which he took on and defeated Juan de Palermo. These competitions, genuine intellectual tournaments, involved contenders setting a series of problems for their opponents and seeing who was able to solve the greatest number of them in the least time. The only condition was that the proposer of a given problem must be able to solve it. One of the problems described by Fibonacci is as follows: Find the number such that adding or subtracting 5 to its square gives a square number in both cases. Curiously, 1225, the year in which the book is published, is a perfect square (the previous is 1156 and the next, 1296), the only square year that Fibonacci would live through.

In the same era as Fibonacci, the learned Arab, Ibn Kallikan was the first to tell the famous legend of the inventor of chess: *The History of Sissa Ben Dahir and the Indian King Shirham* (1256). According to the legend, Sissa, the inventor of chess, succeed in entertaining King Shirham in such a way that the king agreed

to give him any gift he desired. Sissa asked the king for one grain of wheat for the first square of the chess board, 2 for the second, 4 for the third, 8 for the fourth, and so on, each time doubling the number of grains until reaching the 64th square. The king thought Sissa's request to be very small, until he realised that it could never be met. Effectively, $2^0 + 2^1 + \ldots + 2^{62} + 2^{63} = 2^{64} - 1 =$ 18,446,744,073,709,551,615, more than 18 trillion, and more grains than the world's annual production of wheat.



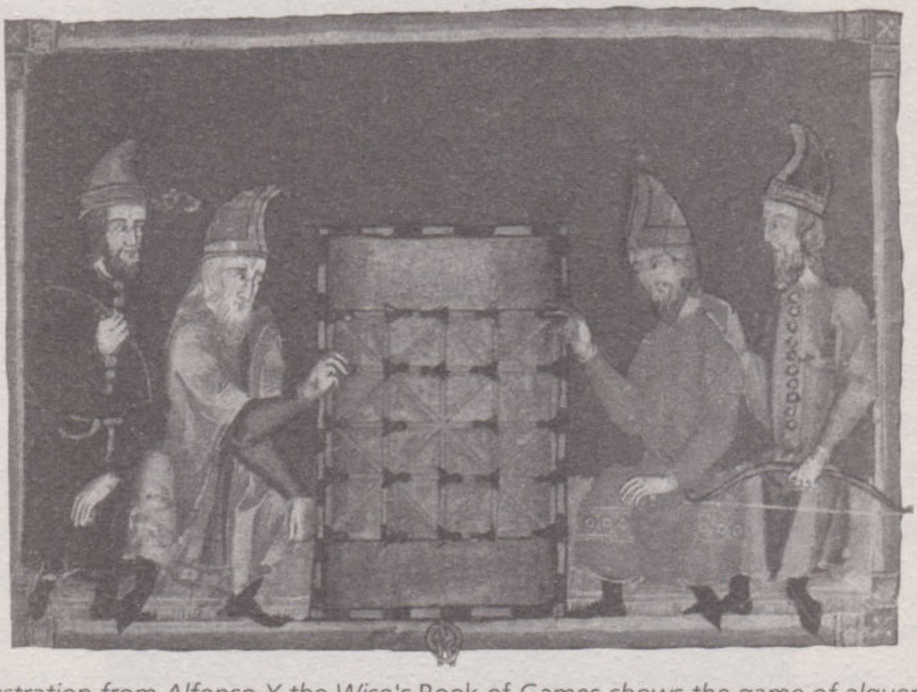
A page from Fibonacci's Liber Abaci.

Also in the 13th century, specifically in 1283, Alfonso X the Wise's *Book of Games* was published. Although the book is more centred on games than on mathematical aspects, the analysis of the games is interesting as it gives an idea about the types of games (both of chance and strategy) being played at that time, and the level of knowledge shown by the winning strategies being suggested. In addition to chess and various games of chance, the book describes *Alquerque*, the oldest known strategy game, that is to say, a game without the element of chance.

ALFONSO X THE WISE'S *BOOK OF GAMES*.

In 1283 King Alfonso X the Wise of Castile commissioned a text known as the *Book of Games*, also referred to as the *Book of Chess, Dice and Tables*. The book consists of 98 pages with 150 colour illustrations and covers the main table games of the time, including chess, alquerque, dice games and board games, a family that includes backgammon.

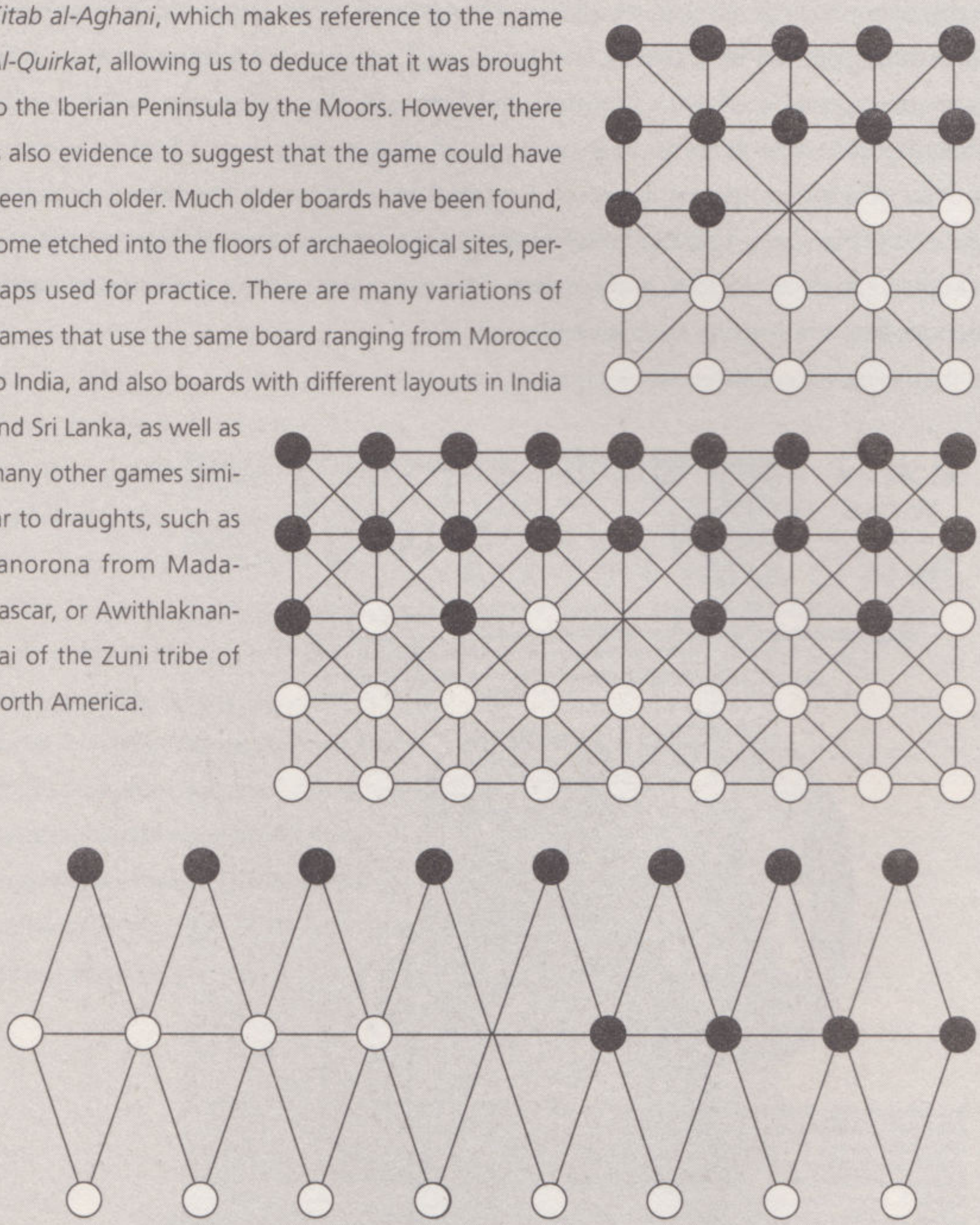
The only original copy that has been saved is held in the library of the El Escorial Monastery. The book, the West's oldest book on games, is of enormous value, both on account of its content, which provides us with details of the games played on the Iberian Peninsula around 800 years ago, and the magnificent illustrations it contains.



An illustration from Alfonso X the Wise's Book of Games *shows the game of alquerque.*

ALQUERQUE, AN ANCIENT STRATEGY GAME.

Alquerque is the name of a two player game described in Alfonso X's *Book of Games.* It is played on a 5 x 5 square board with 12 counters for each player. The counters are arranged so the central square is empty. The aim of the game is to remove the opponent's counters. The manner of doing this shows clearly that this game was the predecessor of draughts. The oldest written reference to the game dates back to an Arabic manuscript from the 10th century, the *Kitab al-Aghani*, which makes reference to the name *Al-Quirkat*, allowing us to deduce that it was brought to the Iberian Peninsula by the Moors. However, there is also evidence to suggest that the game could have been much older. Much older boards have been found, some etched into the floors of archaeological sites, perhaps used for practice. There are many variations of games that use the same board ranging from Morocco to India, and also boards with different layouts in India and Sri Lanka, as well as many other games similar to draughts, such as Fanorona from Madagascar, or Awithlaknannai of the Zuni tribe of North America.



From top to bottom, the starting position for Alquerque, Fanorona and Awithlaknannai.

Mathematics and games during the Renaissance

The mathematics of the Renaissance period is mainly represented by a group of mathematicians known as the Italian algebraists, including Tartaglia, Cardano, Bombelli, Ferrari and del Ferro, whose main contributions lay in the field of algebra, particularly in solving equations. In the realm of mathematics and games, two names stand out in particular: Tartaglia and especially Cardano. The self-taught maths teacher Nicolo Fontana (1499-1557), known as Tartaglia ('The Stammerer'), is famous for his discovery of a general method for solving third-degree equations. Similarly, he was the first person to translate the works of Euclid and Archimedes into Italian. His mathematical dual with Scipione del Ferro, in the same style as the medieval tournaments – which he won by solving all the problems that had been set, the majority of which required the resolution of third-degree equations – would appear to be the reason why Cardano asked him for the formula for solving the equations. Tartaglia agreed to reveal it, and Cardano wasted no time in publishing the method, much to the wrath of its true inventor.

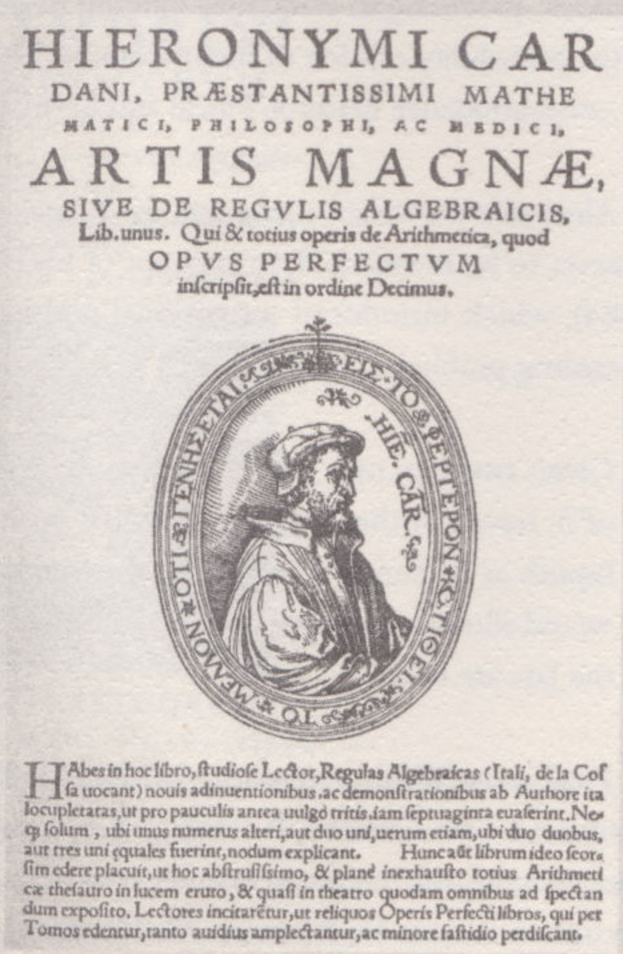
QVESITI,
ET INVENTIONI
DIVERSE
DE NICOLO TARTAGLIA,
Di nouo restampati con vna Gionta al sesto libro, nella quale si mostra duoi modi di redur vna Città inespugnabile.
La diuisione, & continentia di tutta l'opra nel seguente foglio si trouarà notata.

The cover of Nicolo Tartaglia's Quesiti et Inventioni Diverse *(1546).*

GEROLAMO CARDANO (1501-1576)

Doctor, mathematician, astronomer, astrologer and game-player (just a few of his many interests), Cardano formed part of a group of mathematicians that included Tartaglia, del Ferro, Ferrari and Bombelli, which contributed to the development of algebra in 16th century Italy. The details of his life are well known thanks to his autobiography, *De Vita Propria*. In contrast to many of his contemporaries, Cardano achieved some fame and notoriety while living, especially as a doctor. Very much a Renaissance figure, he was interested in many things, making use of reason in his attempts to make progress in all facets of the knowledge of his time, although in many instances, his work is marred by a high degree of naivety, irrationality and even superstition. As a result, while he was often a brilliant figure, this brilliance is often highly contradictory.

Among his most significant mathematical works, we can find the *Ars Magna*, (1545), a key text of Renaissance algebra. Prior to this, in 1539, he had written another book entitled *Practica Arithmetica*. Similarly, he also wrote one of the first books on games and mathematics, the *Liber de Ludo Aleae* (*Book on Games of Chance*), which tackled problems related to the probabilities of dice games for the first time, providing ingenious, if occasionally incorrect, solutions. The book was written by Cardano in around 1564. However it was not published until one century later, with the publication of his complete works. The work should be considered as the first to discuss probability, but it did not have the same impact as that of Pascal and Fermat, whose correspondence is considered by many as the start of probability theory.



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib.unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inſcripſit,eſt in ordine Decimus.

HAbes in hoc libro, ſtudioſe Lector, Regulas Algebraicas (Itali, de la Coſ ſa uocant) nouis adinuentionibus, ac demonſtrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam ſeptuaginta euaſerint. Neq; ſolum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo ſeorſim edere placuit, ut hoc abſtruſiſſimo, & planè inexhauſto totius Arithmeticæ theſauro in lucem eruto, & quaſi in theatro quodam omnibus ad ſpectandum expoſito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore faſtidio perdiſcant.

Frontispiece of Gerolamo Cardano's Ars Magna.

Although Tartaglia did not specifically analyse games of chance in the same way as Cardano, he did publish a book, *Quesiti et Inventioni Diverse* (1546), which contains puzzles and problems, some of which are still well known to this very day, such as:

> A man has 17 horses and bequeaths them to his three sons in the proportion 1/2, 1/3 and 1/9. How are the horses shared?

> A man has three pheasants and wishes to share them between two parents and two children in such a way that each receives a pheasant. How can this be done?

However, without doubt one of the first mathematicians who attempted to analyse games of chance with some degree of formality was Cardano, perhaps the most brilliant and versatile mathematician of his time, although his work related to games was not published until a century after it was written, meaning that it did not have the impact it deserved. At first glance, it was the first book to state the so-called 'problem of points', providing an erroneous solution based on the points of each player and not on the probability of each of them winning. This was one of the problems studied in the correspondence between Pascal and Fermat and will be discussed in Chapter 3.

Alongside the Italian algebraists, the French mathematician Nicolas Chuquet also deserves to be mentioned on account of his book *Triparty en la Science des Nombres* (1484), which introduced recreational problems and contained the first so-called 'decanting problems', one of which is as follows:

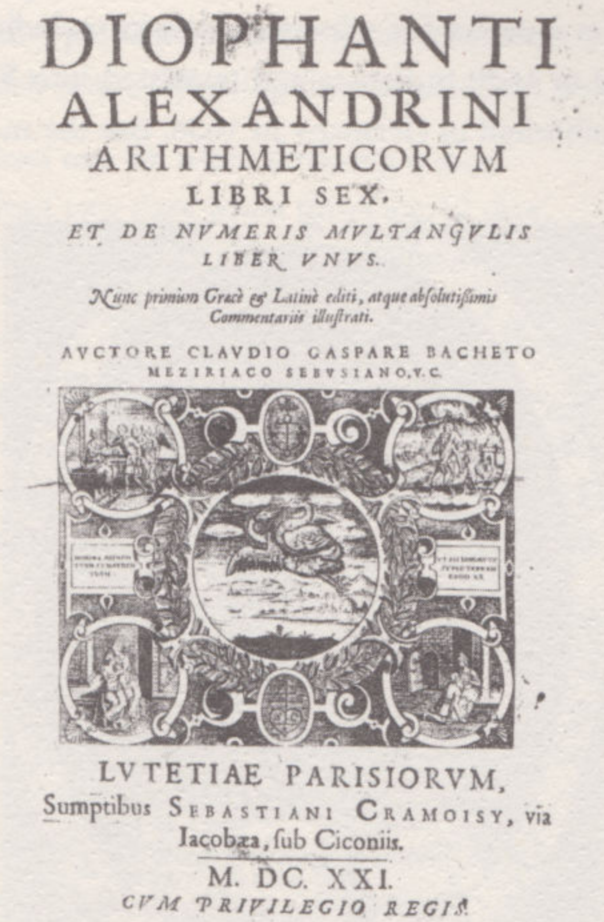
> Given two jars, one with a capacity of 3 pints and another with a capacity of 5, how can the larger jar be filled with exactly 4 pints, transferring the liquids as required, if we know that neither of the jars has a marking that would allow us to know any volume other than that which is indicated when the jars are full.

Finally, the Welsh mathematician Robert Recorde (1510–1558) should also be mentioned. Like Cardano, Recorde led an eventful life and like many other Renaissance scientists was interested in other areas of thought such as astronomy and medicine. Recorde is famous for his work *The Whetstone of Witte* (1557), which was the first to make use of the = sign, with Recorde remarking that nothing is more equal that two

parallel lines. Although in our times it is difficult to imagine algebra without using this symbol, there was a long period before it was adopted universally, coexisting alongside others such as *AE* (the start of the word *aequo*, meaning 'to equal') until the 18th century. The work also contains recreational problems that are largely solved using algebra.

Mathematical games from the 17th century to the present day

As we have seen, serious and playful mathematics have coexisted since the origins of the science, but it is not until the 17th century that we see the emergence of mathematical puzzles as an independent discipline. As mentioned in the previous section, the first large-scale study exclusively dedicated to recreational mathematics, *Problèmes Plaisants et Délectables qui se Font par les Nombres*, written by Claude-Gaspar Bachet de Méziriac (1581–1638), was published in 1612. The mathematician, poet, translator and one of the first members of the Académie Française is also known for



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.

LVTETIAE PARISIORVM,
Sumptibus Sebastiani Cramoisy, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

The cover of the Latin edition of Diophantus' Arithmetica *with comments by Bachet de Méziriac.*

his book of puzzles and for his annotated Latin version of Diophantus' *Arithmetica* (1621), originally written in Greek. Fermat wrote his famous conjecture in the margin of a copy of this book , but that will be discussed in Chapter 3.

The rise of mathematical puzzles: the 17th and 18th centuries

The work of Bachet de Méziriac is a compendium of the mathematical puzzles of the time, including known puzzles, such as 'the wolf, the goat and the cabbage', magic squares, whole number questions, and weight-based problems, such as the following: find the minimum number of weights and their respective values in order to determine the weight of an object whose value is between 1 and 40 on a set of scales with two plates.

From that point on, the 17th century saw the appearance of a range of similar works. In 1624 Henry van Etten, pseudonym of the French Jesuit Jean Leurechon, published *Récréations Mathématiques*, a similar book to Bachet's only more successful. It served as a model for subsequent books including those by Claude Maydorge, published in 1630 in France and translated into English in 1633, and Daniel Schwenter, published in Germany in 1636. But the most influential work

A portrait of the mathematician and linguist Daniel Schwenter.

was Ozanam's, *Récréations Mathématiques et Physiques*, revised and expanded by the mathematician and science historian, Jean E. Montucla in 1725.

In the 18th century, William Hooper's work *Rational Recreations* (1774) is also worthy of mention. The book contains the *Vanishing Paradoxes*, a good example of how a seemingly simple puzzle can require the use of interesting mathematical properties.

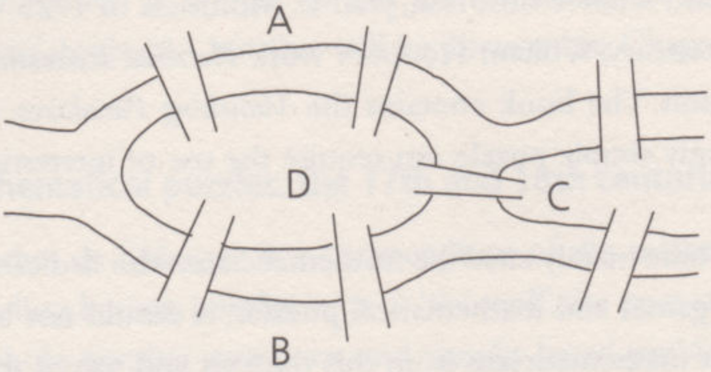
Although we have mainly cited the mathematicians, who dedicated specific works to the world of games and mathematical puzzles, it should not be forgotten that many other great mathematicians from this time set and solved recreational problems that would go on to become classics of the genre. Isaac Newton (1642–1727), Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855) are three of the best known figures.

In his work, *Arithmetica Universalis*, published in Latin in 1707, Newton introduces basic recreational problems alongside his contributions to mathematics. The most famous of these is the problem known as 'Newton's cows'. It is included below as an example of a probability problem related to games of chance. A number of unloaded dice are thrown, which of the following three possibilities is most likely?

a) Obtaining at least one six when throwing 6 dice.
b) Obtaining at least two sixes when throwing 12 dice.
c) Obtaining at least three sixes when throwing 18 dice.

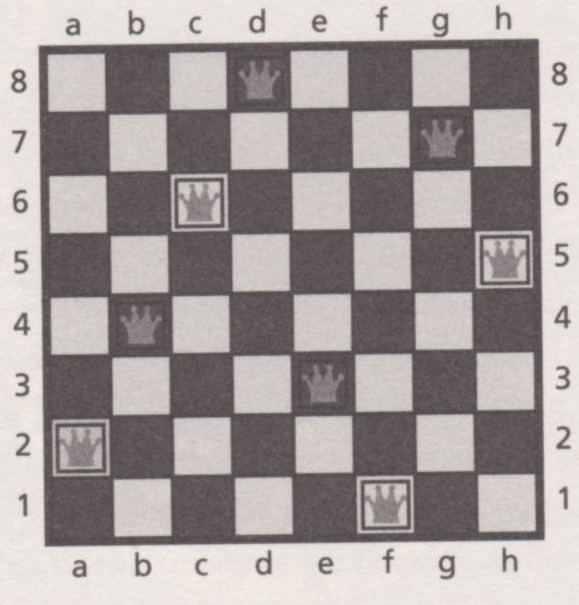
The reader will have no difficulty in solving the problem after reading and working through similar problems that are discussed in Chapter 3.

Euler, perhaps one of the most prolific of mathematicians, wrote a number of playful studies, such as the Graeco-Roman squares, also known as 'Euler squares', in the field of combinatorics. These are a sort of magic square in which the n symbols must be laid out in an $n \times n$ grid in such a way that each symbol appears in every row and column. These can be viewed as genuine predecessors to the Sudoku problems that are currently popular. However, undoubtedly the most famous of his recreational problems is the 'Seven Bridges of Königsberg', which he published in Latin in 1759 in the proceedings of the Berlin Academy of Sciences, and which is the origin of graph theory. (A graph is a graphical representation of a relationship between the elements of a set. It is composed of vertices – elements of the set – and edges which join them – the relationships between the elements). Graph theory is mainly used in stating and solving optimisation problems.



The problem of the Seven Bridges of Königsberg asks whether it is possible, starting on any of the four parts of dry land, to find a route which crosses over each of the bridges only once. Euler showed that no such route exists and discovered the conditions that made it possible to determine if such a path is possible or not.

Finally, Gauss, alongside his great contributions to mathematics, dedicated a small part of his time to the study of recreational problems, including the 'eight queens problem': place eight queens on a chess board such that none of them threatens the other and then find the number of different solutions. Then generalise the problem for *n* queens on an *n* x *n* board. By first using an intuitive method that is then systematised to create a problem of permutations, Gauss verifies that the eight queens problem has 92 different solutions.

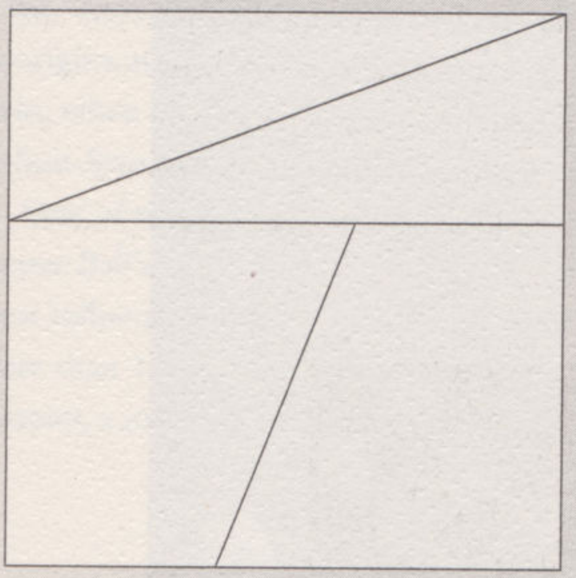


This 8 x 8 board shows just one of the many solutions to the "eight queens" problem.

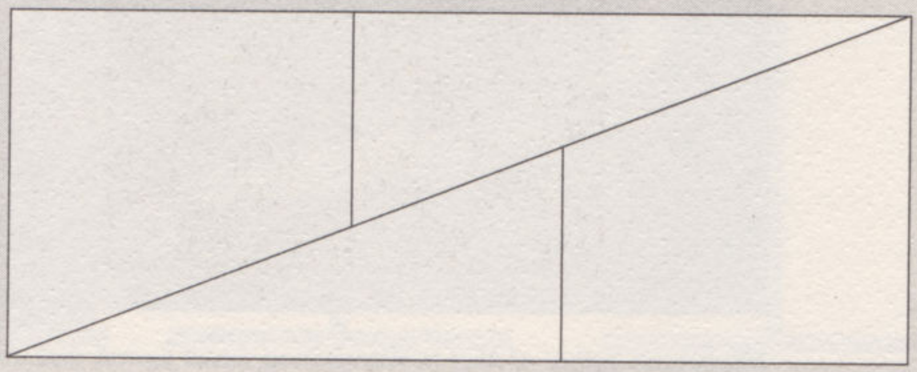
HOOPER'S PARADOX

In this puzzle, an 8-unit long square is divided into two triangles and two trapezoids. The four parts are then used to make a rectangle 5 units wide and 13 units long. If this is possible, the area of the square (64 units^2) would be equal to that of the rectangle (65 units^2), which would 'prove' that 64 equals 65. It is left to the reader to discover the impossibility of 'covering' the rectangle and just where the 1 unit^2 "hole" is hidden.

Even when the paradox is solved, it remains a mathematical curiosity. It is possible to see further ramifications when analysing the problem in greater depth. By observing the lengths of the various shapes, and putting them in order, we obtain the numbers 3, 5, 8 and 13, numbers from the Fibonacci series. One of the properties of this series is that the square of a term is equal to the product of the previous and following term plus or minus 1, or rather: $a_n^2 = a_{n-1} \cdot a_{n+1} + (-1)^{n+1}$. This explains why taking a square the length of which is a term from the Fibonacci series and a rectangle with sides that are the previous and next terms, can create this paradoxical problem. The paradox is solved and the puzzle correctly assembled if we turn to the golden ratio (Φ), repeatedly related to the Fibonacci numbers. Take a square with length Φ and four parts, as above, and form a rectangle with lengths 1 and $\Phi + 1$. It is now possible to see that the area of the square (Φ^2) is equal to that of the rectangle, which is $1 \cdot (\Phi + 1)$.



Hooper's paradox suggests that with the two triangles and two trapezoids contained in the square can be rearranged to create a rectangle with an area of 1 square unit more.



Mathematical games in the 19th and 20th centuries

Games and mathematical puzzles continued to be developed throughout the 19th century and the first part of the 20th, resulting in an enormous increase in material. Among the authors from the 19th century were James Joseph Sylvester (1814–1897), Lewis Carroll (1832–1898), Édouard Lucas (1842–1891) and Walter W. Rouse Ball (1850–1925). Given that it is not possible to discuss the work of all of these figures in depth, the most relevant aspects are mentioned below, with greatest attention paid to the work of Carroll and Lucas.

The reverend Charles Ludwig Dogson, known as Lewis Carroll, the author of the *Alice in Wonderland* stories, was a mathematician and professor at Oxford University. His great interest in mathematical puzzles led him to plan an uncompleted collection of books with the title of *Curiosa Mathematica*. The second of these, entitled *Pillow Problems*, shows off his ingenuity for solving problems, although their level of difficulty ranges from simple jokes – I have two watches, one stopped and another

The famous author of Alice in Wonderland, *Lewis Carroll, also devised a number of mathematical games.*

a minute slow, which keeps the best time? – to more complex matters – Given three random points on an infinite plane, what is the probability that they form an obtuse angle triangle?

In addition to being an ingenious creator of mathematical and logic games, he was also extremely knowledgeable when it came to language, as is shown by his stories about Alice and the numerous word games he invented. One of these, called 'Word Ladder', consists of passing from one word to another, both with the same number of letters, changing only one letter at a time in such a way that each intermediate word has a meaning. For example, to go from ROOT to NOSE, one possible solution would be: ROOT – LOOT – LOST – LOSE – NOSE.

However, the most important analyst of games and mathematical puzzles at the time was the French mathematician Édouard Lucas, who specialised in number theory. He was particularly famous for his work on the Fibonacci series, and was author of the excellent compendium *Récréations Mathématiques*. The book contains 35 works, some dedicated to the mathematical analysis of games and others which deal with puzzles. Among the original games devised by Lucas is the game known as the 'Towers of Hanoi', which the author himself, in order to plant doubts about its origins, attributed to Claus, an old Chinese teacher from a school called Li-Sou-Stain, when it was published in 1883. Note that Claus is an anagram of Lucas, and Li-Sou-Stain is Saint Louis, the school at which Lucas taught mathematics.

One of the last works of mathematical puzzles of the 19th century is Walter W Rouse Ball's *Mathematical Recreations and Essays* (1892), which became one of the most influential books on recreational mathematics during the 20th century, with more than 12 editions, one of these revised and updated in 1938 by Harold Scott Coxeter, a mathematician specialising in geometry.

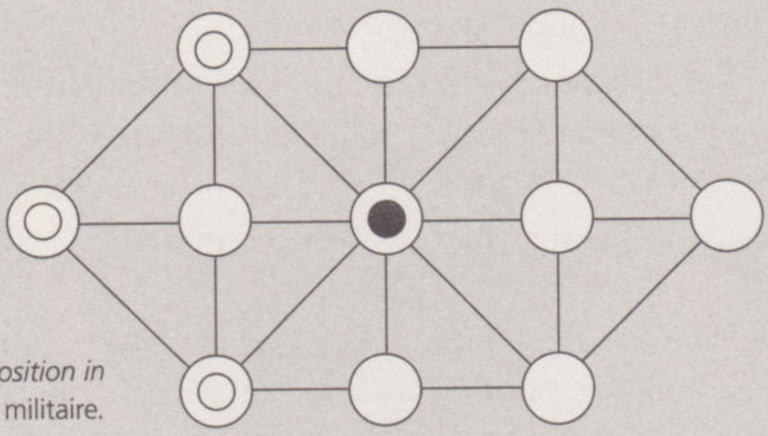
The starting position of the Towers of Hanoi. The aim is to transfer all the discs to another tower, only moving one at a time and without ever placing a larger disc on top of a smaller one.

JEUX MILITAIRE

One of the games analysed by Édouard Lucas in the third volume of his mathematical puzzles was fox and geese This belongs to the group of catching or hunting games and it was highly popular in England from Tudor to Victorian times, but had a 15th-century French origin.

'*Jeux militaire*' is a similar two-player hunting game without the intervention of chance, which was popular in French military circles in the 19th century. One player has three white pawns and the other, who will start the game, has just one black pawn. The pawns are placed on a board with 11 squares (see the layout below for starting position). The aim of the white pawns is to surround the black one, which is trying to escape. Each pawn moves to an empty neighbouring square, following the lines of the board. However, while the black pawn can move in any direction, the white ones cannot go back.

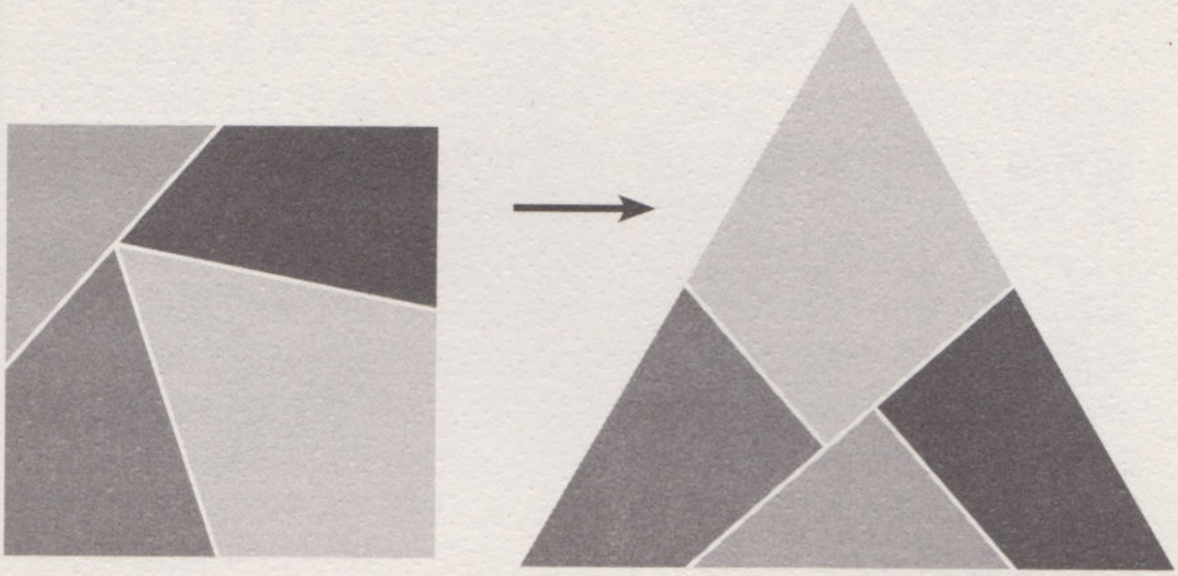
This seemingly simple game is extremely subtle and although it initially appears that the black pawn can always escape, the exhaustive analysis carried out by Lucas shows that there is a winning strategy for the white pawns that always have at least one move to prevent the escape of the black pawn. The analysis of the game shows that a maximum of 12 moves are required, although the game is really reduced to 16 different games. It seems impossible that such a limited game demands such precision from the player moving the white pawns; however, they can always win if they discover how.



The starting position in jeux militaire.

The transition between the 19th and 20th centuries is marked by the work of two of the most prolific authors of all time in the field of mathematical puzzles, the Englishman Henry E. Dudeney (1857–1930) and the American Sam Loyd (1841–1911). Many of the puzzles that are still enjoyed by the general public to this day are collected together in the immense work of these two great authors.

Among other works, Henry E. Dudeney is author of *The Canterbury Puzzles* (1907) and *Amusements in Mathematics* (1917), the latter containing one of the best and most varied collections of mathematical puzzles in history.



Henry E. Dudeney's Haberdasher's Puzzle solves the problem of how to divide a right-angle triangle into four parts to form a square.

Dudeney's great collection of puzzles includes cryptograms, operations in which numbers are indicated by letters and in which each letter can be substituted for a number in such a way that the same letters have the same numbers and different letters different ones. The most famous cryptogram is the one which appeared in a letter Dudeney sent to his father asking for money with the following sum: SEND + MORE = MONEY. The reader must substitute each letter for a number such that the sum indicated is correct (the only solution to this cryptogram is: 9,567 + 1,085 = 10,652).

Sam Loyd published a large number of his problems in the magazines of his time, although it was his son, Sam Loyd, Jr., who brought much of his work together in 1914, a few years after his death, under the title *Sam Loyd's Cyclopaedia of 5,000 Puzzles, Tricks and Conundrums*. Loyd's puzzles include the well-known one in which 9 points must be joined together in the form of a 3 x 3 grid by drawing 4 straight segments without lifting the pen (the same for 16 points, a 4 x 4 grid, and 6 segments), or the many structures upon which certain numbers must be placed in order to meet certain conditions. For example, position the numbers 1 to 8 on the vertices of a cube in such a way that the sum of the 4 vertices of each face is the same.

A page from Sam Loyd's Cyclopaedia of 5,000 Puzzles, Tricks and Conundrums.

The tradition created by Dudeney and Loyd continued throughout the 20th century, and other key authors of the first half of the century include Maurice Kraitchik (1882–1957), who compiled various books of games and was editor of the Belgian journal *Sphinx*. For many years after the Second World War, the field was dominated by another great creator and collector of puzzles, Martin Gardner (1914–2010), author of a wealth of books and articles published over the course of more than 25 years in the popular science journal *Scientific American*. Until shortly before his death, Gardner continued to publish versions of his work, more than 70 books in total, including *Origami, Eleusis and the Soma Cube,* published in 2008. In addition to his own creations, he published some of the most interesting and innovative puzzles of others, including John Conway's 'Game of Life' (1970) and Robert Abbott's *Eleusis* (1956).

Other important 20th-century authors include Yakov Perelman, the principal exponent of what can be referred to as the Russian school, the Frenchman Pierre Berloquin, and the Englishmen Ian Stewart, Brian Bolt and David Wells, all of whom have written a number of books and published their work in various journals. A number of Spanish authors are also worthy of attention who, like the authors above, have tried to bring mathematics to the general public, largely through books

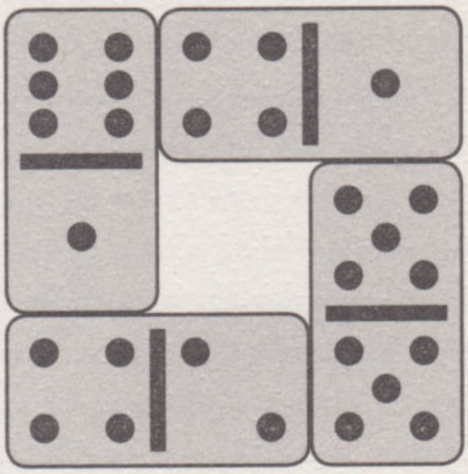
ELEUSIS, ROBERT ABBOTT'S GAME OF GAMES

If a game is defined by an aim and a set of rules, Eleusis is no normal game, since its aim is to guess the rule created by each of the players, which varies from game to game. The game is for 4 to 8 players and is played with three packs of cards and a few counters. It consists of the same number of rounds as there are players. For each round, a different player deals (they become 'god', the creator of the rule). They deal 14 cards to the other players and place a card on the table. Prior to having done so, they will have written down a secret rule, which makes it possible to form a sequence of cards. Examples of extremely basic rules include red-black or even-odd, although there is an infinite number of possible rules: Even after red and odd after black, or four even cards of a different suite and four odd cards of the same suite.

The creator of the rule wants the rule to remain a secret. However it should not be too difficult either, or they will not score many points by playing the game. The other players try to discover the rule (without ever saying it) by taking turns to form a sequence of 'good' cards. God decides if the card is good and will be added to the row, or if it is bad, in which case it will be placed under the last good card, incurring a two card penalty. From the 40th card onwards, playing a bad card means being excluded from the game, which ends when a player finishes their cards or all players are eliminated.

First published in 1963, Abbott's New Card Games *includes the rules for nine original games, including Eleusis.*

and articles on games and mathematical amusements. Some of the most prolific of these include Mariano Mataix, Miguel de Guzmán and Fernando Corbalán. All these figures created and collected an enormous body of work which, when added to the work of the writers already mentioned, constitutes an inexhaustible source of mathematical games and amusements.



A domino problem by Yakov Perelman: Four dominoes have been arranged in a square such that all sides have the same total. The challenge is to create seven squares like this using all the dominoes.

The appearance of game theory

An extremely important part of this book, specifically Chapters 4 and 5, concerns game theory. It is this development that proves the mathematical principle whereby, sooner or later, concepts and models from the science find applications in real world situations, including those arising in an apparently far-removed branch of the discipline, such as the analysis of games.

A good player is one that makes the best decisions when making their moves. The analysis of games sets out to discover precisely what the best moves are and, where possible, determine a way of playing that will always win. This is theoretically possible in finite games in which chance has no role, although in practice the scale of some games, most notably chess, can make it impossible to devise a definitive strategy for wining.

Game theory began with the work of John von Neumann, specifically the book jointly published with the economist Oskar Morgenstern, *Theory of Games and Economic Behaviour* (1944). The book is based on a type of abstract game for two or more players in which the winnings and losses of each player following the moves of the whole group of players are determined beforehand. Generally speaking, the players make their moves simultaneously and do not know the strategy of their opponents. These games, which act as mathematical models, were initially used for

the analysis of competitive situations in the sphere of economics, and the authors described a method to determine the optimal strategies for each player. The success of von Neumann's so-called 'minimax' strategy and its extensions that factored in the effects of chance – referred to as 'mixed strategies' – led the mathematicians and economists to make use of game theory to study more complex situations.

However, what began as a set of applications in the field of economics, initially using highly simplified models, gradually evolved throughout the second half of the 20th century. With the introduction of games in which the winnings of one player were not necessarily the losses of the others, the idea of cooperation was introduced, or rather the tension between conflict and cooperation, generating models of games which were increasingly closer to reality, not only for the science of economics, but also in other fields, such as military tactics, politics, ecology and even philosophy. All these apparently unrelated disciplines shared a requirement to make decisions in situations that can be conceptualised as a game, although the term *game* here loses its aspect of playfulness, instead becoming more focused on the idea of risk. As the formulation of these games gets closer to reality and they become more complex, they allow for more open solutions in which mathematics can share its knowledge together with ideas from other areas such as morals, ethics and philosophy, and in the end, the study of human behaviour.

John von Neumann gives one of his lectures at the American Philosophical Society, an institution of which he was a leading member.

One of the aspects that makes game theory most interesting, along with its results, is the way it can be applied to areas of social science that entail a certain level of chance and in which the variables involved are related to human behaviour, both individually and in groups. Thus, the development of game theory led to the creation of various dilemmas, generally focused on the tension between conflict, risk and cooperation. These can be applied in a wide range of situations and constitute a significant part of this theory. Among the best known, and those that shall be discussed in the final chapter of this book, are the 'prisoner's dilemma' and the game of 'chicken', or its development in terms of the evolution of the species, the 'hawk-dove dilemma'. In a certain sense, these dilemmas show both the difficulties of quantifying human behaviour but also reveal the many ways in which the chaotic hubbub of society could be described with mathematics.

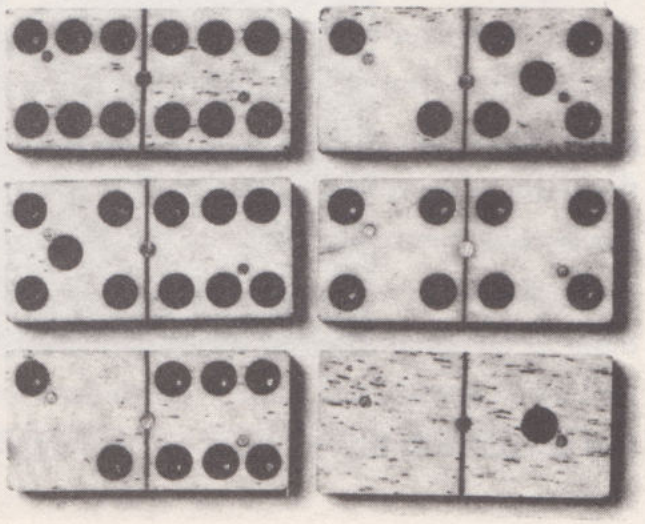
Chapter 2

Strategy Games and Problem Solving

Although there are few things more entertaining than jokes, on account of the challenge they represent in terms of ingenuity and capacity for reasoning, they are not only good for amusement; as J.E. Littlewood has observed, a good mathematical joke is better, and better mathematics, than a dozen mediocre papers.

Martin Gardner

Games can be classified in a number of ways, according to various criteria: the place where they are played, the number of players, the length of a game, level of difficulty, etc. When it comes to mathematics, there is one element that allows us to identify two types of game – chance. This can appear in various ways, as part of the starting conditions of the game, or when making particular moves. By way of example, in the majority of card games, the cards are randomly dealt between different players; this is also the case for dominoes in which the tiles are distributed randomly. In contrast to this, the initial layout of a chess board is pre-determined and is always the same, just like in ludo, backgammon and reversi. In terms of the possible moves, there are many games in which chance does not intervene, with each player being free to decide their move from a range of possibilities. In other games, however, there is an element of chance that is often created by rolling one or more dice, with the player deciding which moves to make according to the result of the dice throw.



Domino tiles from the 19th century. Dominoes is just one of the games in which chance only intervenes when selecting the tiles. For the rest of the time it depends on the players' skill.

The term 'strategy games' is used to refer to the set of games in which there is no element of chance. They are only dependent on the decisions of the players when making their moves. The absence of chance means that such games can be analysed in order to find a winning strategy. In some cases it is possible to define a full strategy for this, while in others, the complexity of the game makes this impossible, although it may be possible to show that such a strategy exists for one of the players. In spite of the apparent diversity of these games and their solutions, only a small set of mathematical techniques and concepts are used in their analysis and these mainly correspond to the areas of arithmetic (systems of numbering and divisibility) and geometry (situations of equilibrium, chiefly symmetry).

The concept of a winning strategy

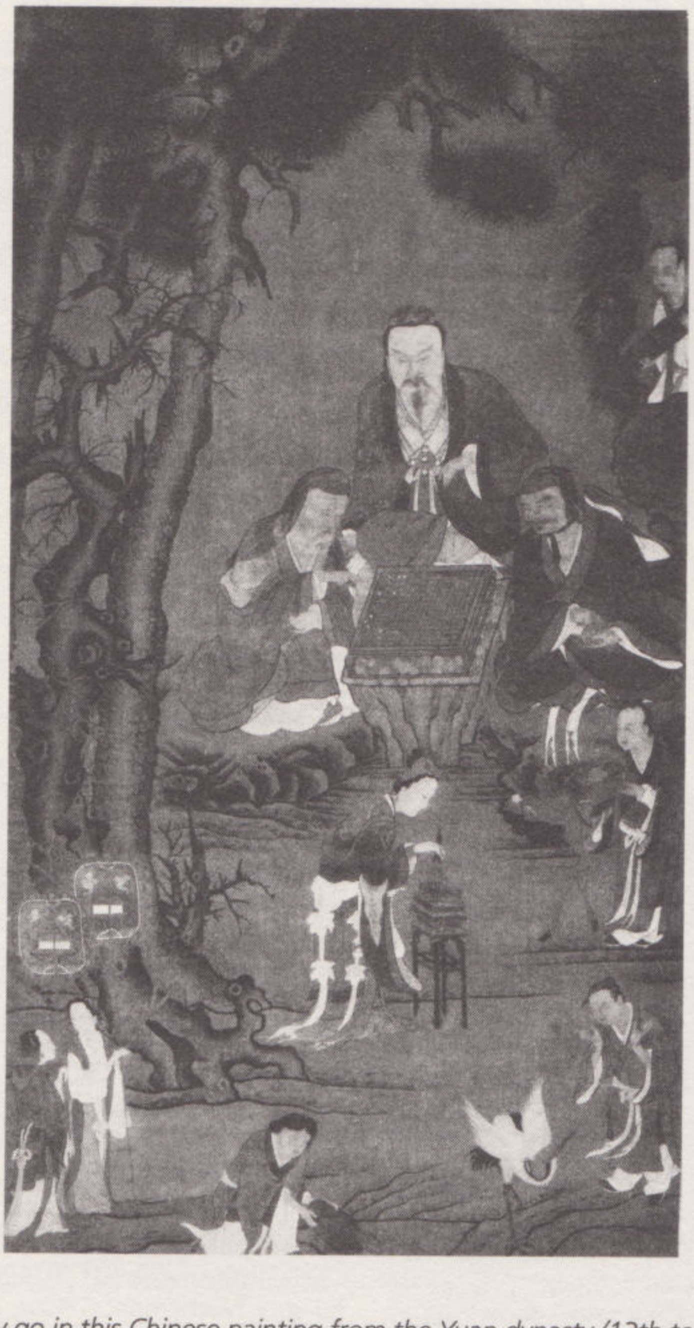
Although in mathematical terms the word 'game' can refer to both games in their own right (those in which there is more than one player, with determined rules and a goal that makes it possible to decide who has won the game) and puzzles, from here on we will discard the latter term in order to focus our attention on games with two or more players. It is possible to classify these games in quite different ways but, from a mathematical point of view, there is an initial classification that makes it possible to distinguish two main types: finite games and games in which there is an element of chance. In this chapter, the former group will be referred to as 'strategy games' and the latter as 'games of chance'.

When a game is played and its workings are well known, the question arises of how to play to win every time. In games of pure chance (such as snakes and ladders) the above question is absurd, since the movements of the players' counters are dependent on the numbers on the die and the application of rules according to the squares on which the counter is positioned; or rather, there is no possibility of making decisions, meaning there are no better or worse moves. The result of a game of this type depends entirely on luck, and as such, analysis of the game – from the point of view of finding winning strategies – is not possible. In this respect, it can be said that mathematically speaking, the game is of no interest.

At the other end of the scale are finite information games. At any given moment in the game it is possible to know all possible moves and their consequences (at least in theory) and there is no element of chance. In our culture, the game that best encapsulates this idea is chess, although many more strategy games are known,

both traditional ones (mancala, draughts, noughts and crosses, etc.) and more modern creations (hex, Nim, reversi, abalone, etc.).

The concept of a 'winning strategy' arises when discussing the analysis of one of these games. This is a set of conditions that allows one of the players (they are normally only two-player games) to decide how to play at a given moment in time, taking into account the moves played by their opponent, in order to win, regardless of the other's moves. The existence of a winning strategy assumes that the game ends



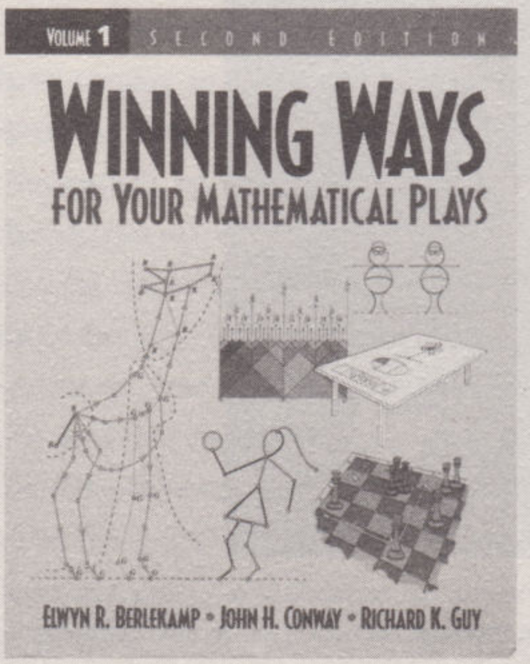
Three people play go in this Chinese painting from the Yuan dynasty (13th to 14th centuries).

with one of the players winning, although this does not always happen in games that can also end in a draw, such as chess. In this case, it should be noted that there is not so much a strategy for always winning, but rather not losing. When a strategy game cannot end in a draw, it is possible to determine whether there is a winning strategy for the first or second player, according to the properties of the game. However, this does not necessarily mean that it is possible to formulate this strategy, since finding it depends on the complexity of the game.

THE BIBLE OF WINNING STRATEGIES

Possibly the most extensive and relevant book on strategy games is the four-volume *Winning Ways for your Mathematical Plays* (1982), written by four eminent 20th-century mathematicians: Elwyn Berlekamp (1940–), professor of computer science at the University of California, Berkeley from 1971; John Conway (1937–), who has published work on finite set theory and is professor at the Universities of Cambridge and Princeton, and creator of the so-called 'game of life', a game that simulates cellular life on a computer; and Richard Guy (1916–), emeritus professor at the University of Calgary. The features of the games covered in the book are as follows:

1. Games for two players who make alternate moves.
2. Games with a starting position and which have a finite number of moves.
3. Complete information games where the players know all the possible moves they can make at a given moment in time.
4. Games in which chance does not intervene either at the start or during play.
5. Games in which the progress of a game does not allow the repetition of moves and in which moves are determined in such a way that a player who is unable to move loses.



Front cover of the first volume of Winning Ways for your Mathematical Plays, *by Berlekamp, Conway and Guy.*

Suppose that a two-player game has the following properties:

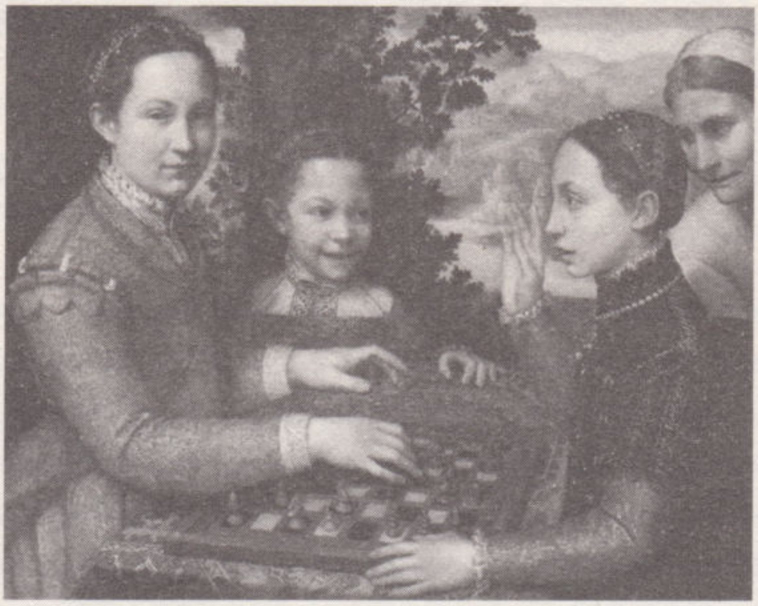
1. It is a complete information game, that is to say that at any given moment, each player has complete information to decide how to make their move.
2. Players take turns to make alternate moves.
3. There is no element of chance in the game.
4. Each game ends after a finite number of moves, with one of the players winning.

Under the conditions above, it is possible to show that there must be a winning strategy for one of the players, either the first (player A) or the second (player B). Let us assume that A does not have a winning strategy, or in other words B always has a move available to which A will be unable to find an appropriate response. This means that A will lose and B will win, making it possible to claim that there is a winning strategy for B. However, although this argument means it is possible to claim that in this type of game there is always a winning strategy, this is not to say that it can be easily determined, only that it is theoretically possible.

For games that do not necessarily have a finite number of moves, the extension of this result depends on the acceptance of the so-called 'axiom of choice'. This famous and controversial mathematical axiom states that given a collection (finite or infinite) of non-empty sets with no elements in common, it is possible to form a new set by selecting a given element from each of the sets in the collection. In 1930 Banach, Mazur and Ulam used this axiom to define a non-finite game and proved that there was no winning strategy for either A or B.

Exploiting advantages, defining strategies: Nim games

If we go back to considering the classification of games and focus on what we have called strategy games, it is possible to distinguish two types – those with properties and rules that are simple, with a short duration and a limited quantity of information, referred to as 'small-scale strategy games'; and those such as chess or go, in which complete control is practically impossible on account of the duration of the game, the complexity of the rules and above all the high number of possible moves in any given situation. By studying some small-scale strategy games it is possible to see how mathematics can be used to analyse games to reveal the player who has an advantage and how to determine their winning strategy.



The Chess Game, *a canvas painted in 1555 by the Renaissance painter Sofonisba Anguissola. Chess is a game without an element of chance, but in which the number of possible moves is so high that so far mathematicians have not solved all of possible versions of the game.*

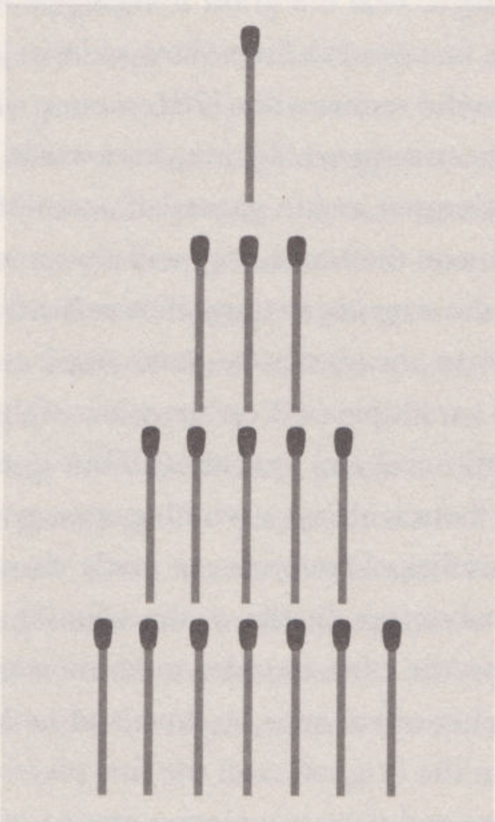
The relationship between games and mathematics can refer to different aspects of games, as we have seen in the first chapter, and mathematics is particularly useful for determining the winning strategy in strategy games. A strategy game is very similar to a mathematical problem and its solution in that both require knowing how to act in order to be successful, to be the winner. Winning is equivalent to solving the maths problem. As such the determination of winning strategies requires the use of heuristics (for example, working backwards assuming the result of the game, the application of symmetry, the establishment of analogies with another game that has already been solved, etc.) similar to those used to solve mathematical problems. This means that when a winning strategy is known for a game, the latter stops being an object of play and becomes a solved problem. Obviously this only happens for specific games, those where playing quickly goes beyond entertainment and enters the realm of occasionally sophisticated mathematical theories the study of which is discussed below.

A small group of two-player strategy games, known as Nim games, consists of placing one or more piles of counters on the table and establishing rules for how they can be removed. The aim of the game is either to remove the last counter, or the opposite, to force the other player to remove the last counter. Both the origin

of this type of game, which has been traced back to the Far East, and its etymology are unknown. Among the variety of suggestions, the word *nim* meant to 'steal from' or 'rob' in Old English; it has also, rather ingeniously, been observed that applying rotational symmetry to the word NIM gives the word WIN. However, the analysis for finding a winning strategy for this type of game was first published in 1902 by C.L. Boston, a mathematician from Harvard University, making the game over one hundred years old.

The game acquired a certain degree of popularity in Europe during the 1960s thanks to French filmmaker Alain Resnais's film *Last Year in Marienbad* (1961), in which two characters repeatedly play a version of the game. This led to the version in question – details of which will be provided further on (game 5) – becoming known as Marienbad, the name of the small spa town in the Czech Republic where the film is set.

Finding a general winning strategy for solving any Nim game is one of the best examples of the application of mathematics in the analysis of games, specifically the effectiveness of representing numbers in a binary system.



Marienbad is a variant of the Nim games.

Towards the definition of a strategy

Below, we begin by analysing games with a single pile of counters in which it is possible to remove a variable number of these in a single move, the minimum number being one and the maximum being n. To do so, two specific cases are stated and then a general explanation is given. The simplest of these games is as follows.

Game 1 (two players): 20 wins

20 counters of the same colour are placed on the table and each player takes it in turn to remove one or two counters. The player who removes the last counter wins the game. Which of the two players, the first or the second, has the advantage? How can they ensure they always win? If the number of counters is changed does this stay the same? And what if the aim is changed so that the person who removes the last counter loses? This is a simple enough game so as to be completely analysed, determining the winning strategy and generalising it for any number of counters. If the reader is not familiar with the game, before going any further it might be of interest to find a partner and play it a few times in order to answer the above questions.

Any player will soon discover that the player who leaves 3 counters on the table will win on their next turn. This is a good thing to know but does not make it possible to win, since we first need to know how to leave just three counters. Now we know that the player who removes the 17th counter will win, meaning that the number of counters has been reduced. Working backwards, it can be observed that if 6 counters are left, they also win, and in general, if one of the players always leaves a multiple of three counters on the board, they will always win the game. This makes it possible to determine the winning strategy: if there are 20 counters in the starting position, the first player can always win by removing 2 counters in the first game and then always leaving a multiple of 3 on the table (if the second player removes one counter, the first removes 2 and vice versa). Thus in this game, the first player has the advantage, since there is always a winning strategy available to this player.

Varying the initial number of counters can partly change the strategy and even the player who has the advantage. In fact, as the winning strategy consists of leaving a multiple of three on the table, in order to know what will happen, it suffices to divide the initial number of counters by three and look at the remainder of the division: if it is two (as in the original case) the first player will win by removing 2 counters in the first game and then completing groups of three (if their opponent takes 1, the first player must take two, and vice versa); if the remainder of the division

is 1 (e.g. starting with 19, 25, 100 or 2,011 counters), the first player also wins, this time removing just one counter in the first move. However, if the remainder is 0 (the number of counters is divisible by 3), the second player will always win by removing 2 counters when the first player removes one and vice versa. In this case, it will never be possible for the first player to leave a multiple of 3 counters on the table.

Thus, we have succeeded in generalising the game for any initial number of counters. It can be made more general still if the number of counters that can be removed in each game is varied.

Game 2 (two players): 100 loses

The first player writes a number from 1 to 10 on a piece of paper. The second thinks of a number from 1 to 10, adds it to the number written by the first player and writes the total on the paper. The game continues, with each player adding a number from 1 to 10 to the previous result. The player who reaches a three-figure number (100 or more) after adding their number, loses the game. What is the winning strategy? Which of the two players, the first or the second, has the advantage? What happens if the rules of the game or its aim are changed?

As has previously been suggested, it is a good idea to play a few games first in order to discover the winning strategy for one of the players and also to think about the relationship between this game and the previous one. In order to analyse the game in a way that allows us to discover a winning strategy, we can proceed as follows: If the first person to reach 100 loses, the person who manages to write 99 wins. What number must they have written previously to ensure to ensure it is possible to reach 99? The answer is 88, since this forces their opponent to write a number between 89 and 98, making it possible to reach 99 the next time round. As in the previous case, working backwards (giving the numbers 88, 77, 66…, all the way down to 11), it is clear that the winning strategy involves making groups of 11. Thus it is now possible to formulate the winning strategy: The player that writes 11 and then successive multiples (if the opponent adds n, the winner should add $11 - n$) will reach 99 and win the game. Given that it is impossible for the first player to reach 11 on their first move, a winning strategy exists for the second player. As in the previous game, it is the first player who always wins when the target number is not a multiple of 11, and the second wins when it is.

Game 3 (two players): Complete generalisation

Let us suppose that there are m counters on the table and in each move, between 1 and n ($n < m$) counters can be removed. The player who removes the last counter wins the game. Which of the players, the first or the second, has a winning strategy? What is that strategy? If the aim is changed, and the player who removes the last counter becomes the loser, how does the strategy change?

Actually, this is not just a single game, but a set of abstract games, of which the two previous examples are just specific cases. Hence, the winning strategy for this game is a generalisation, which solves an infinite number of games at the same time. The formulation of this strategy is as follows: Divide m by $n + 1$ and determine the remainder of the division that will be a number between 0 and n. Hence we have two cases:

a) The remainder of the division is 0. In this case, there is a winning strategy for the second player who must leave a multiple of $n + 1$ on the table; in order to do so, if the first player removes p counters ($0 < p < n + 1$), the second must remove $n + 1 - p$ counters, something which is possible since the number is always between 1 and n.

b) The remainder of the division is r ($0 < r < n + 1$). In this case, there is a winning strategy for the first player who must remove r counters on their first turn, leaving a multiple of $n + 1$ on the table, such that they may now play as if they were the second player and apply the winning strategy in case A, that is to say if the second player removes p counters ($0 < p < n + 1$), the first must remove $n + 1 - p$ counters.

This solution solves an infinite number of specific games. By means of example, it can be applied to the following game: there are 2,010 counters on the table and each player can remove between 1 and 49. Which player has the winning strategy? And what is it? If the game is defined such that the player removing the last counter loses instead of winning, it suffices to note that all that is required to win is to remove the second-last counter, leaving just one on the table. In this case, the strategy will remain the same, although the number of counters will now be $m - 1$, instead of m.

All these games, which start with a single pile of counters can be thought of as a simplification of the so-called game of Nim, to which we shall now turn out attention.

A complex strategy: The game of Nim

It is possible to further generalise the previous group of games by allowing the number of piles of counters to be any finite number. The so-called game of Nim consists of starting with various piles of counters that can have different sizes. The rules of the game allow each player to remove as many counters as they like on their turn, with a minimum of 1 and a maximum of all the counters in a single pile. The winner of the game is the player who manages to remove the last counter, although it is also possible to play in such a way that the person removing the last counter loses.

Game 4 (two players): First version of Nim

Let us start with three piles of counters, with 1, 3 and 5 counters in each pile. Each player takes it in turns to remove the desired number of counters from a single pile (a minimum of 1 counter, a maximum number of all). The player who removes the last counter wins the game. Which player has the winning strategy?

The analysis of the game allows us to see that in this case there is a winning strategy for the first player, although of all the possible initial moves, only one guarantees their victory. In fact, if the reader practices the game, they will discover that neither player should make the following moves:

a) Leave two piles with the same number of counters.
b) Remove all the counters from a pile.

In fact, if player A makes move a), player B can remove the counters from the third pile and win the game by copying their opponent's moves. If A removes *n* counters from one of the piles, B removes the same number of counters from the other pile, such that when A removes the last counter from one pile, B will remove the last from the other and win). Similarly, if A makes move b), B will then remove counters from the pile which has the most to leave two piles with the same number of counters, winning the game by following the same strategy explained in the previous case. As such the player who forces their opponent to make one of the 'prohibited' moves wins the game. In the case we are considering here, if the first player removes 3 counters from the pile of 5, leaving three piles of 1, 2 and 3 counters, respectively, they will win the game, since this means their opponent will either have to clear a pile or make two piles equal (with a total of 1 or 2 counters).

It is clear that this strategy is too specific and not easily generalised for a variable number of piles, or even for three piles with different and larger numbers of counters. Yet mathematics can help us determine a completely general strategy that can be used for any number of piles with any number of counters in each pile. To do so, it is necessary to note – as shown in the following examples – that if the quantity of counters in each pile is expressed in base two (the binary system) and the numbers are written down in such a way that the units corresponding to each number are arranged in a column, then the parity of at least one of the columns will be altered in each move. This is because a move means it is necessary to change only one of the digits of one or more columns and at least one of the digits will go from 1 to 0. Thus, if the sum of all the digits of each column is even in the starting position, there is a winning strategy for the second player (which consists of leaving all the columns with an even sum after their first move, something the first player cannot do). On the other hand, if at least one column has an odd total, the first player will have a winning strategy since they will be able to leave all the columns with an even total after their first move.

To better understand how this strategy works, let us consider a few examples of how it can be applied to specific cases. First with three piles of 1, 3 and 5 counters respectively (game 4 which we have previously solved), and then with a more common version of the Nim games, Marienbad, which begins with four piles of 1, 3, 5 and 7 counters.

In the first case, as we have noted, there are three piles with 1, 3 and 5 counters, respectively.

1 in base two:			1
3 in base two:		1	1
5 in base two:	1	0	1

Adding the units of each column confirms that all have an odd sum (3, 1 and 1, from right to left, respectively). In this case, there is a winning strategy for the first player. To win, they must play in such a way as to leave all the columns with an even sum. The only way to do this is to change the number 5 (101) to 2 (10) by removing 3 counters from the pile with 5. Now we have:

1 in base two:			1
3 in base two:		1	1
2 in base two:		1	0

All the columns now have an even total, meaning that any move made by the second player will leave one of the columns with an odd total, allowing them to leave the column totals with an even number until the final position (all numbers will be 0, that is to say, all the columns will add up to an even number).

Game 5 (two players): Marienbad

Four piles of counters are placed on the table, with 1, 3, 5 and 7 counters in each. The players take turns to remove as many counters as they wish from a single pile (minimum one counter, maximum all). The player who removes the last counter wins the game. Which player has the winning strategy?

Proceeding as in the previous case, we now have:

1 in base two: 1
3 in base two: 1 1
5 in base two: 1 0 1
7 in base two: 1 1 1

Since the total of all the columns of the numbers expressed in binary notation is even in the starting position, the first player cannot win and the winning strategy is there for the second player to exploit. Effectively, any move made by the first player will leave at least one column with an odd number. Let us suppose that they remove a counter from the pile of three. This will leave:

1 in base two: 1
2 in base two: 1 0
5 in base two: 1 0 1
7 in base two: 1 1 1

NIMROD

At the start of the 1950s, engineers working for the British company Ferranti designed the first computer created exclusively for playing games. Its name was NIMROD, with the first three letters corresponding to the game of NIM, since this was precisely the game that the creators programmed into the computer. The panel of the computer had a series of lights, which lit up to represent the positions of the game. The prototype was presented at the *Festival of Britain* in 1951 and is credited with ushering in the era of electronic games.

The second player must now change a number so that adding up the column on the right gives an even number (and the others remain the same since their total is even), or rather they must remove just one counter from any of the piles except the second one, which in binary notation would result in changing a 1 to a 0 in the right-hand column.

Although the Nim strategy is a lot harder to reveal than the ones for the previous games, there is a general scheme that is used to determine the winning strategies for all these games. Find an equilibrium consistent with the end situation of the game that can be reached by one of the players and not the other. Thus in the first game in this chapter (in which 20 wins), the equilibrium consists of leaving a multiple of 3 counters on the table. In the second game (in which 100 loses), it is writing a multiple of 11. In the final examples (Nim), the equilibrium is leaving a number of counters on each pile, which when written as as columns of binary numbers add up to an even total.

Nim games are often presented in their opposite form, i.e. the player who removes the last counter loses the game instead of winning. In this case, the same player who would normally win still wins and the strategy is initially the same, only varying when the 'normal' move (which would lead to winning the first version) leaves all the piles with at least 2 counters. Now the winning move consists in leaving an odd number of piles with only one counter, instead of an even number that would be the correct move in the normal game.

Once the winning strategy for any Nim game is known, the question arises of whether it is possible to create a game of the same type in which there is not a winning strategy. The answer is yes, and it leads us to so-called 'Nimbus' games. These are based on Nim games with the following condition imposed: more than one counter may only be removed from a given pile when the counters are connected – that is to say, there are no gaps caused by a previous move. This introduces a condition related to the position of the counters within each row, something that has not been taken into account thus far. It is the same as saying that every time counters are removed from a row, the row can be split into two (something which will always happen when the counters removed do not include those at the end of the row), creating new piles and changing the game in such a way that it is no longer possible to apply the strategy used for Nim games.

THE SYSTEM OF BINARY NUMBERS

Binary numbers are a positional numbering system which allows any number to be expressed using the two digits 0 and 1. To convert a binary number into a decimal one, all that needs to be done is to swap each 1 for the power of two the exponent of which corresponds to the position. In other words, the digit on the right is equal to 2^0, the following to 2^1, the next to 2^2 and so on. For example, in decimal form the base two number 110101 would become: $1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 + 1 \cdot 2^5 = 1 + 4 + 16 + 32 = 53$.

Similarly, to write a decimal number in binary form, the number should be divided by 2, then the quotient by 2, and so on until the quotient is 1. The last quotient is the first digit on the left and the remainders of the divisions, from last to first, are the digits that follow (the remainder of dividing by 2 can be only 0 or 1).

As an example, 39 in base two is: 100111, since 39 / 2 gives 19 (remainder 1); 19 / 2 gives 9 (remainder 1); 9 / 2 gives 4 (remainder 1); 4 / 2 gives 2 (remainder 0); 2 / 2 gives 1 (remainder 0). The idea is to express these numbers as the sum of the successive powers of 2; thus, $39 = 1 + 2 + 4 + 32 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5 = 100111$ in base two.

Although the concept of binary is a relatively new one, the property on which it is based (namely that all numbers can be expressed as the sum of the powers of two different numbers) has been known and applied throughout the ages. For example, the system used in ancient Egypt for multiplication consisted of doubling one of the two terms and dividing the other by two (if the number is odd, the previous number is divided by two), something that always worked as a result of this property.

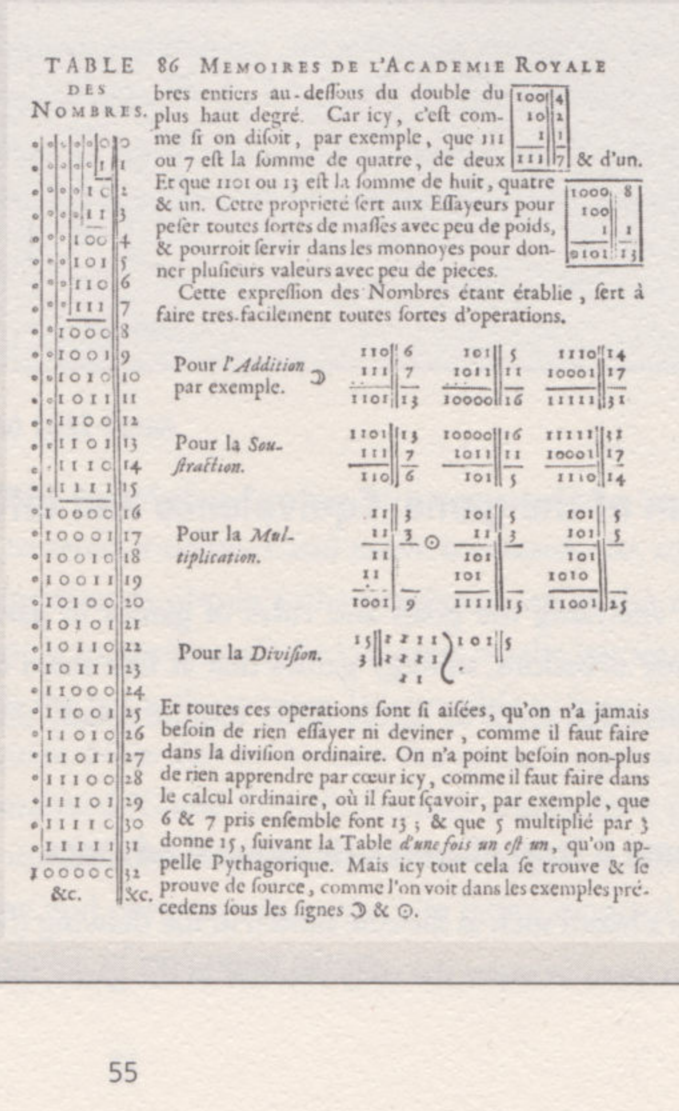


TABLE DES NOMBRES. 86 MEMOIRES DE L'ACADEMIE ROYALE

0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15
10000	16
10001	17
10010	18
10011	19
10100	20
10101	21
10110	22
10111	23
11000	24
11001	25
11010	26
11011	27
11100	28
11101	29
11110	30
11111	31
100000	32
&c.	&c.

bres entiers au-deſſous du double du plus haut degré. Car icy, c'eſt comme ſi on diſoit, par exemple, que 111 ou 7 eſt la ſomme de quatre, de deux & d'un. Et que 1101 ou 13 eſt la ſomme de huit, quatre & un. Cette proprieté ſert aux Eſſayeurs pour peſer toutes ſortes de maſſes avec peu de poids, & pourroit ſervir dans les monnoyes pour donner pluſieurs valeurs avec peu de pieces.

100 | 4; 10 | 2; 1 | 1; 111 | 7

1000 | 8; 100 | ; 1 | 1; 1101 | 13

Cette expreſſion des Nombres étant établie, ſert à faire tres-facilement toutes ſortes d'operations.

Pour *l'Addition* par exemple. ☽

110 | 6 + 111 | 7 = 1101 | 13; 101 | 5 + 1011 | 11 = 10000 | 16; 1110 | 14 + 10001 | 17 = 11111 | 31

Pour la *Souſtraction.*

1101 | 13 − 111 | 7 = 110 | 6; 10000 | 16 − 1011 | 11 = 101 | 5; 11111 | 31 − 10001 | 17 = 1110 | 14

Pour la *Multiplication.* ⊙

11 | 3 × 11 | 3: 11, 11 = 1001 | 9; 101 | 5 × 11 | 3: 101, 101 = 1111 | 15; 101 | 5 × 101 | 5: 101, 1010 = 11001 | 25

Pour la *Diviſion.*

15 | 1111 ; 3 | 11 : 101 | 5

Et toutes ces operations ſont ſi aiſées, qu'on n'a jamais beſoin de rien eſſayer ni deviner, comme il faut faire dans la diviſion ordinaire. On n'a point beſoin non-plus de rien apprendre par cœur icy, comme il faut faire dans le calcul ordinaire, où il faut ſçavoir, par exemple, que 6 & 7 pris enſemble font 13; & que 5 multiplié par 3 donne 15, ſuivant la Table *d'une fois un eſt un*, qu'on appelle Pythagorique. Mais icy tout cela ſe trouve & ſe prouve de ſource, comme l'on voit dans les exemples précedens ſous les ſignes ☽ & ⊙.

A page from the Mémoires de l'Académie Royale des Sciences *dedicated to the binary system developed by Leibniz in 1703.*

SOME GAMES TO PRACTISE

Turning a die This is a strategy game for two players. The first player places the die on the table, selecting a number, which is left facing upwards. The next player gives the die a quarter turn, leaving another number on the top and adding it to the previous term. The players take it in turns to turn the die (they can obtain any number except for those on the top and bottom), adding the number on the upper face to the previous total. The player who gets to 31 wins the game. Which player has the advantage? How can they ensure that they always win?

Cutting the rectangle. A strategy game for two players. A 17 x 15 rectangle is drawn on a sheet of squared paper (using the lines of the grid, i.e. one unit is the length of a square). The square in the lower right corner is marked. The players take turns to divide the rectangle into two parts by drawing a straight vertical or horizontal line (using the lines of the grid) and eliminating the part that does not contain the marked square. The player who is unable to divide the rectangle (i.e. the player who is left with just the square that was marked) loses the game. Which player has the advantage? How can they ensure that they always win?

Crossing the circle. A strategy game for two players. Draw a circle on a sheet of paper and mark eight points on it at random. The players take turns to join two points by drawing a line. They can join any points they like, provided they have not yet been connected. However the line they draw may not cross any of the previously drawn segments. The player who is unable to draw a new segment loses the game. Which player has the advantage? If the number of starting points is altered, does the advantage stay with the same player?

Aim of the game: Equivalence and difference

By analysing the goals and rules of games in parallel, it is possible to see that on many occasions, strategy games that at first seem different are in fact the same, and games that seem to hardly differ require totally different strategies.

Game 6 (two players): Hexagonal advance

On a board such as the one shown in the drawing (figure 1, opposite), each player takes it in turns to move the only counter in the game, initially placed at S. They must move

it to a neighbouring square, always moving towards the right either horizontally or diagonally. The player who manages to place the counter on the last square (position M) wins the game.

Figure 1

If the reader tries to solve the game, he or she will easily find the tiles on which the counter must be positioned in order to win. Working backwards, they will arrive at the conclusion that the first player has a winning strategy if they position the counter on the marked tiles. It is not immediately obvious that this game is equivalent to Game 1 (in which 20 wins) until we see that each move is an advance of 2 (always continuing in the same row) or by 1 (when changing rows). A numbered list of the tiles makes this clearer (figure 2).

Figure 2

Game 7 (two players): Add the last item

Three counters are positioned on a board that consists of six squares in a single row. The players take it in turns to select a counter and move it towards the right by as many squares as they like (minimum of one and a maximum up to the end of the board). The aim of the game is to position all the counters on the last square on the right. The player who adds the last is the winner. It is possible to have more than one counter on the same square. Note that in this case the game is equivalent to the first version of Nim we analysed (Game 4): each counter represents a pile and moving it towards the right means removing counters from the pile in such a way that the counter reaching the end corresponds to the pile being emptied.

Let us now analyse the equivalence of two other games.

Game 8 (two players): Tsyanshidzi

Two piles of counters are placed on the table (e.g. 7 and 5 counters in each pile). The players take it in turns to remove as many counters as they like from one of the piles (minimum of one). They can also remove counters from both piles, although in this case, the number of counters removed must be the same for both piles.

Game 9 (two players): Save the queen

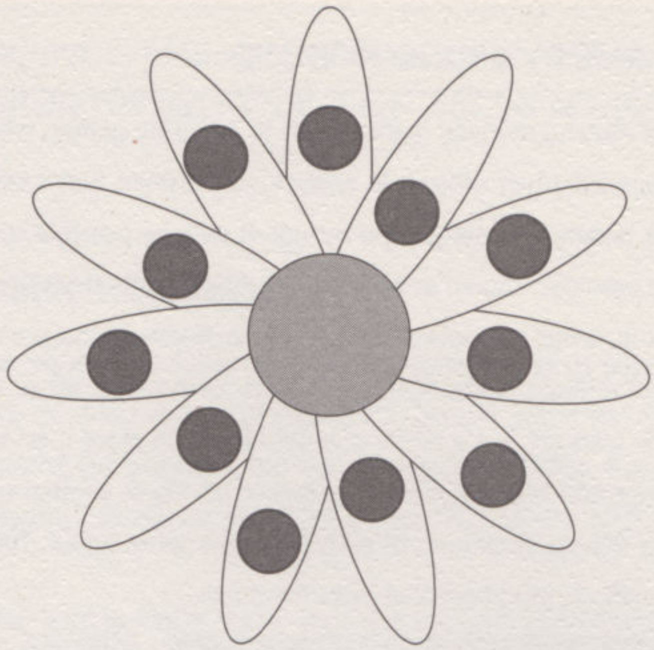
A queen is placed on any square on a chessboard (e.g. H8). Each player takes it in turns to move the queen as many squares as they like to the left, downwards or diagonal (left and down). The player who manages to position the queen in the square A1, that is to say the intersection of the first row and the first column wins the game.

The first of these games, Tsyanshidzi, is a Nim game that allows the possibility of removing counters from more than one pile, something that we have not yet considered, and which considerably complicates the process of determining a generalised winning strategy. Analysing the moves allowed in the second (Game 9) quickly reveals its equivalence to the first, transforming the movements of the queen into the removal of counters. Moving the queen in a row is equivalent to removing counters from the first pile; moving it in a column is the same as removing counters from the second pile; and moving it diagonally is the same as removing the same number of counters from both piles.

The previous examples have shown that on certain occasions, games which at first sight look different are actually very much the same. To be able to see this, it suffices to be able to transform the goals and rules of one of the games into the other. However, on other occasions we find the opposite. Games that seem practically the same are in fact highly different, especially if we focus on their winning strategy. Let us consider the following game that at first looks identical to Game 1 (in which 20 wins).

Game 10 (two players): The daisy

A daisy is drawn with 11 petals, and a counter is placed on each one. The players take it in turns to remove one or two counters, however, they may only remove two counters if they are positioned together (i.e. they belong to neighbouring petals).



Starting position of the daisy.

The game is extremely similar to the first one we analysed in this chapter (in which 20 wins), but with 11 counters instead of 20. In line with this, the first player can win the game by removing 2 counters on the first move and completing groups of three. However the restriction that is imposed – two petals may only be removed if they are side by side – completely invalidates the previous strategy. Now what really matters is the position of the counters and the number becomes irrelevant. In fact the initial number of counters does not matter, since as long as it is greater than 3, the winning strategy can be formulated in the same way for any quantity.

Strictly speaking, the game is not a Nim game, but belongs to the group of games known as Nimbus, whose general strategy is unknown. In fact, it is a simple case of such games. In this specific example, it is possible to see that the second player can always win for any initial number of counters by using a strategy of symmetry. By playing the game it is possible to observe that if a player manages to separate the petals into two groups with the same configuration, they can easily win the game playing symmetrically, by always removing counters symmetrically matching the ones removed by their opponent. Given that in their first move, the first player cannot separate the counters into two groups – that would mean removing two counters that were not together – and will thus leave a gap. The second player can make another gap that will separate the counters into two groups.

BABYLON, A GAME BY BRUNO FAIDUTTI

The current world of abstract strategy games tends to generate games, which in spite of their apparent simplicity, are extremely difficult to analyse, to the point that it can be almost impossible to determine a winning strategy, even though it may be possible to establish that one exists. The following example, known as Babylon, a strategy game devised by the French game designer Bruno Faidutti, is one example of such as game. Twelve counters are placed on a table. The counters are four different colours, three of each. Each player takes it in turn to select a pile (to start with the piles are just a single counter high) and place it on top of another pile, subject to the following conditions: a pile can be placed on top of another only when they have the same height or if the upper counter of each pile is the same colour. The first player who is unable to place one pile on top of another loses the game.

Although at first sight it would appear that the game can be solved, perhaps by studying specific cases and trying to extend them, an exhaustive analysis using a computer shows that it is not possible to find a strategy that can be memorised by a human player.

Babylon, a game designed by Bruno Faidutti.

Games and pseudo-games

There are some games that appear similar to those we have already seen, but which in reality cannot be classified as strategy games, because none of the players can intervene in such a way as to change the course of a game. In other words, the winning strategy is contained in the rules of the game in such a way that the decisions made by the players are irrelevant, since they cannot change the result of a game. Such games are often referred to by mathematicians as 'pseudo-games'. Instead of searching for a winning strategy that does not exist, it can be shown that the result

of the game is independent of the players' decisions. The rules and aim of the game determine which of the two players will always win. Let us consider three examples of pseudo-games.

Game 11 (two players): Only odd

20 counters are placed on the table and the players take it in turns to remove 1, 3 or 5 counters. The player who removes the last counter wins the game. Which of the two players, the first or the second, has the advantage? What happens if the number of counters is changed? Is this a strategy game like those before, or is something else happening?

Playing the game will quickly show that the second player wins all the games and that there is nothing the first player can do to win. In fact, the second player is even forced to win, even when they do not want to. In contrast to the previous games, in this game, parity (both in terms of the initial number of counters and the number which can be removed) is decisive. This means that in this case it is not possible to speak of a winning strategy since the solution to the game is determined by its rules.

In fact, if there are initially 20 counters (or any even number) and the first player removes 1, 3 or 5 counters (or any other odd number), the number of counters remaining on the table will always be odd (even minus odd gives odd). When their turn comes, the second player must also remove an odd number of counters, leaving an even number of counters on the table (odd minus odd gives even). As such, the first player will always leave an odd number of counters on the table and the second player will always leave an even number. Since 0 is an even number, the second player will always win, regardless of the moves made by each of the players. Similarly, if the initial number of counters is odd, the first player will always win.

Game 12 (two players): Circles and squares

A series of circles and squares are drawn, arranged in a line. The players take it in turns to remove two similar shapes (two circles or two squares) and replace them with a circle, or remove two different shapes and replace them with a square. As the number of shapes decreases, we are left with just one shape: if it is a square, the first player wins; if it is a circle, the second. Is there a winning strategy? What happens if the initial number of circles and squares is changed? Is this really a strategy game? Consider the initial configuration shown in the drawing overleaf.

After playing various games with this configuration, it will become clear that the second player always seems to win (the last shape is a circle). Changing the number of circles does not appear to change the result, while changing the number of squares does.

To see that this is not in fact a game, since the winner is determined by the initial configuration and the rules, it is necessary to analyse how the number of squares

MORE GAMES TO PRACTISE

Closing a triangle. A strategy game for two players. Draw a circle and mark six points on it at random. The players take turns to join two points by drawing a line. One of the players uses a black pen and the other a red one. Each player can join any two points, provided they are not already joined. The player who manages to draw a triangle with three sides that have the same colour wins the game. Which player has the advantage? How can they ensure that they always win? If the initial number of points is modified does this stay the same? The game can also be played with the opposite goal, i.e. the player who ends up drawing a triangle in their colour loses the game. What happens this time?

The chocolate bar (I) A chocolate bar is made up of 28 square pieces arranged in four rows of seven. The first player breaks the bar into two parts without breaking any of the pieces. The second player takes one of the two parts (discarding the other) and breaks it again. The players take turns to choose one of the parts and break it in two, following the lines that separate the chunks. The first player who is unable to break the bar loses the game. What is the winning strategy? What happens if the bar is made up of 27 pieces arranged in 3 rows of 9.

The chocolate bar (II) Like the previous game, a bar of chocolate is broken in two, however this time it is made up of 50 square pieces arranged in 5 rows of 10. The players take turns to break the bar (or one of its parts) following a vertical or horizontal line (without breaking any of the chunks). This time, none of the parts are removed, all of them remaining in play. The first player to break off a single piece loses the game. What is the winning strategy? What happens if the first player to break off a single piece wins instead of losing?

changes throughout the game. For each move it is possible for the number of squares to remain the same (if two circles are changed for a circle, or a square and a circle for a square) or decrease by two (if two squares are changed for a circle). This implies that if the initial number of squares is even, it will be even for the duration of the game and it is impossible for a square to remain at the end, whereas if it is odd, a single square will be left at the end.

This chapter has focused on strategy games, particularly those that can be analysed in their entirety, in order to see how mathematics is used in determining a winning strategy for one of the players, where one exists. Heuristics such as studying specific cases, assuming the game to be over and working backwards, using symmetry and focusing on parity, all belong in the realm of mathematical problem solving. Nevertheless, when such techniques for analysing games turn up a winning strategy, the game stops being a game and becomes another solved problem.

Broadly speaking, the games that have been analysed correspond to Nim games, which are determined by the number of counters, and Nimbus games in which, in addition to the number of counters, there are also positioning factors that prevent the use of problem-solving strategies applied to the first category and in general make the determination of strategies more complicated.

Chapter 3

Games of Chance

Where do games end and serious mathematics begin? For many, mathematics, deadly boring, is far removed from games. On the other hand, for the majority of mathematicians, it never ceases being a game, although it can be many other things too.

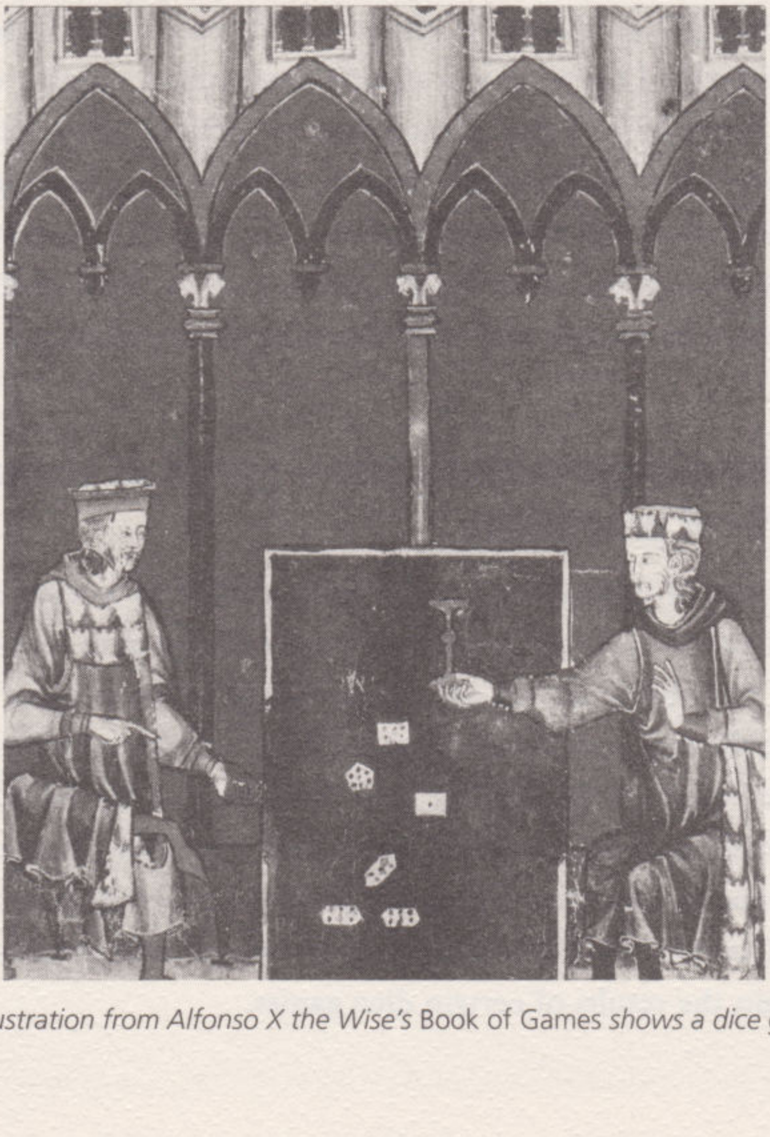
Miguel de Guzmán

This chapter is focused on the connection between games and probability, a relationship that appeared as soon as humanity attempted to model or predict something seemingly as chaotic as chance. Prior to this breakthrough, mathematicians had focused on what was already determined, regular, what could be guaranteed. It could be said that the creation of methods for calculating chance ushered in a new era in mathematics, with its relevance gradually growing as we began to discover more and more applications, a process that continues to this day, when not only chance, but other uncertainties, such as chaos or the irregularity of fractals, came to be studied and modelled using mathematics.

The man who did not want to lose – games of chance and the birth of probability

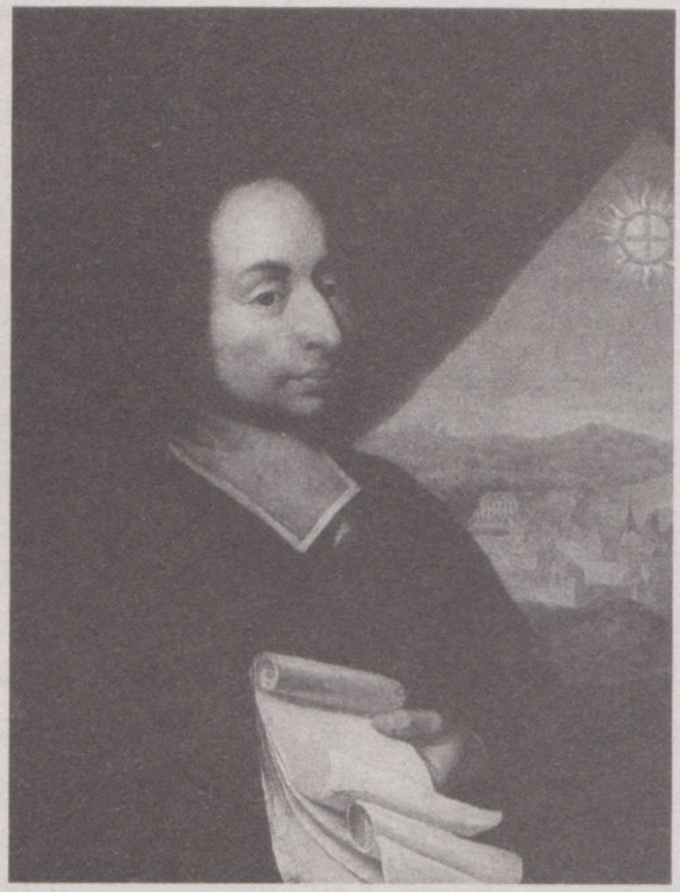
Complex theories of probability are currently applied in a wide range of fields because in our world uncertainty is much more relevant than certainty. However the origins of probability theory are closely linked to wanting to win games of chance. In fact, the origin of the mathematical formulation of a theory of chance, based on the concepts of probability, arose in France in the middle of the 17th century, specifically with the correspondence that took place between Blaise Pascal and Pierre Fermat in 1654 concerning the problems posed by Antoine Gombauld, known as the Chevalier de Méré. This character was an avid gambler who wanted Pascal to explain the results of certain dice games.

The Chevalier de Méré (1607–1685), dedicated a significant part of his life to playing and analysing games of chance using intuitive arguments that often proved to be correct – by chance. It would appear that he won a respectable sum of money betting on seemingly balanced games (in other words, those games that have the same chance of winning or losing). One of the games thought to be 'balanced' at the time was getting at least one 6 when throwing 4 dice, but which Méré knew to have favourable odds. However, he proposed a new game that consisted of getting at least one double six when throwing a pair of dice 24 times, thinking that this would be equally as profitable as the previous one. It did not take long for him to see that this was in fact not the case, and that the supposed advantage actually worked against him, leading him to ask Pascal, around the year 1654, to explain why his reasoning was incorrect and why the new game, unlike the first, was a losing one.



An illustration from Alfonso X the Wise's Book of Games *shows a dice game.*

BLAISE PASCAL (1623-1662)



In spite of his relatively short life, this French philosopher, mathematician and scientist made a number of significant contributions to science and knowledge. He was a child prodigy who, at the age of eleven, was already taking part in the scientific meetings organised by Marin Mersenne. In 1640 he published his *Essai Pour les Coniques* and in 1649 verified Torricelli's findings on atmospheric pressure.

In 1642 Pascal had already designed a machine for calculating so he could help his father, a tax collector in Normandy. The machine, known as the 'Pascaline', was one of the first mechanical calculators that actually worked. Some examples are still conserved in a number of science and technology museums. The calculator, designed for commercial arithmetic, attracted the interest of figures as diverse as Queen Christina of Sweden and the philosopher G.W. Leibniz, who went on to perfect it.

Based on the problems of chance proposed by the Chevalier de Méré, Pascal's correspondence with Pierre Fermat formed the basis of the beginnings of a theoretical formulation for the calculation of probabilities, which Pascal referred to as the 'geometry of chance'. In particular, five of the best known letters, all dating from 1654, analyse the games of chance which had aroused the interest of Cardano.

Another example of his work in this area, *Traité du Triangle Arithmétique* (1654), analyses and proves the properties of the arithmetical triangle, known as Pascal's Triangle, the numbers of which correspond to the combinatorial values, later used by Newton to determine the binomial coefficients.

His mathematical and scientific work came to an end in 1655, when he retired to a convent to dedicate the remainder of his life to his philosophical and religious work.

PIERRE DE FERMAT (1601-1665)

Fermat was one of the most important mathematicians in history, although his dedication to the science was always as an *amateur*. He never managed to publish his work during his lifetime. Instead, it has become known thanks to the correspondence he maintained with the great mathematicians of his time, such as Descartes, Mersenne and Pascal.

Fermat studied law and spent a large part of his life working in Toulouse, where he reached the distinguished position of lawyer in the city's parliament, allowing him to make use of his free time to study mathematics, his real passion. His main interest and the field in which he made his most significant contribution, was number theory. One of his conjectures (that the equation $x^n + y^n = z^n$ does not have whole number solutions for $n > 2$) remained unproven until the 20th century. He also made important contributions to geometry and to the determination of the limits of functions for solving optimisation problems prior to the development of differential calculus. His 1654 correspondence with Pascal is the first known relevant attempt to establish the concept of probability.

Chance tamed: the mathematical study of probabilities

As an introduction to the concept of probability and its basic properties, let us start by analysing the two games played by the Chevalier de Méré. A concrete formulation of the first is as follows: what is the probability of getting at least one six when throwing a die four times? A basic principle of probability theory can be used to solve this problem, where the probability of either an event or its opposite occurring is 1. Thus we must first calculate the probability of not getting a six when throwing a die four times. It is clear if a die is thrown, p (*not* 6) = 5/6. When a die is thrown four times, each throw is independent of the others, meaning that it is possible to determine the combined probability by multiplying the probabilities of each event, making the probability

$$(5/6) \cdot (5/6) \cdot (5/6) \cdot (5/6) = (5/6)^4 = 625 \;/\; 1{,}296 = 0.482 < 1/2.$$

It now follows that the probability of at least one six appearing is:

$$1 - (625 \;/\; 1{,}296) = 671/\; 1{,}296 = 0.518 > 1/2.$$

As such, we have shown that it would be profitable to bet on obtaining a six in four throws of a die, as the Chevalier de Méré had supposed.

It is possible to analyse and solve the second problem in a similar manner: what is the probability of obtaining a double six if a pair of dice is thrown 24 times? As before, we must first calculate the probability of failing to obtain a double six in 24 throws. Throwing two dice, p (no double 6) = 35/36. Thus for 24 throws, we have:

$$p \text{ (no double 6)} = (35/36)^{24} = 0.5086.$$

Based on this result, it is clear that the probability of throwing at least one double six will be:

$$1 - 0.5086 = 0.4914 < 1/2.$$

The games we have just seen are considered to be the first probability problems solved in history. We have already made use of the collection of definitions and properties that form the basis of probability theory.

Achilles and Ajax playing dice on one of the most famous black ceramic Athenian amphoras. It dates to the 6th century B.C. and is just one piece of evidence about the great age of the game.

PIERRE SIMON LAPLACE (1749-1827)

Laplace was one of the great mathematicians of the 18th century. He studied theology and mathematics and taught at the Royal Military Academy in Paris and the École Normale Supérieure. He was a member of the French Institute and the Royal Society of London. During the French Revolution, he contributed to the establishment of the decimal metric system. Under the command of Napoleon, he was a member and chancellor of the senate and received the Legion of Honour in 1805. With the Bourbon restructuring, Laplace became a strong defender of Louis XVIII, who awarded him the title of marquis in 1817.

His main work was in physics and mathematics, and perhaps his greatest contribution to science, is the five volume *Traité de Mécanique Céleste*, published between 1799 and 1825, in which he completed the earlier work done by Newton, Halley and Euler on universal gravity and the proof of the stability of the solar system

From 1780, Laplace worked on probability, publishing his main work, *Théorie Analytique des Probabilités* (1812), which is considered to be the first work in the field. The success of this work led him to write his *Essai Philosophique sur les Probabilités* (1814), which can be considered as a simplified version of his analytic theory of probability. The work contains the most complete and consistent argument for a deterministic conception of the universe. In this respect, Laplace himself claimed: "It is seen in this essay that the theory of probabilities is at least only common sense reduced to calculations (...). there is no science more worthy of our meditations, and that no more useful one could be incorporated into the system of public instruction."

These properties, many of which are dealt with in the aforementioned correspondence between Pascal and Fermat and subsequently established in Laplace's work on probability, are stated opposite, along with examples of relevant dice throwing games.

Event		Probability
1	For any event E, the following condition always holds: $0 \leq p(E) \leq 1$	If a die is thrown, the probability of obtaining a given number between 1 and 6, for example 5, is 1/6, since there are six possible events, only one of which is desirable (i.e. that a 5 comes up).
2	If E is guaranteed $p(E) = 1$ and if E is impossible, $p(E) = 0$	If a die is thrown, the chance of throwing a 7 is 0 (the event is impossible), while the probability of throwing a whole number greater than 0 and less than 7 is 1 (the event is guaranteed).
3	$p(\text{not } E) = 1 - p(E)$	If a die is thrown, p (getting a 6) = 1 – p (not getting a six 6). If a die is thrown four times, we have: p (getting at least one 6) = 1 – p (not getting a 6).
4	If A and B are different events, $p(A \text{ or } B) = p(A) + p(B)$	If a die is thrown, p (getting an even number or getting a 5) = p (even number) – p (5) = 1/2 – 1/6 = 2/3.
5	If A and B are independent events $p(A \text{ and } B) = p(A) \cdot p(B)$	If two dice are thrown, not obtaining a 6: p (no 6 in two throws) = p (no 6) · p (no 6) = 5/6 · 5/6 = 25/36.

THE PROBLEM OF POINTS

We shall now look at an example of one of the earliest probability problems: Rohan and Penny are playing a betting game in which the first person to reach 10 points wins. In each round, the players have the same possibility of winning and the winner receives a point. At the end of the seventeenth round, Penny is winning by 9 points to 8. The game is then interrupted and since neither of the players has managed to reach 10 points, the pair decide to split the money from the bets that have been made so far. How is this done?

The 'correct' solution to the problem may depend on aspects that are not strictly speaking mathematical, meaning that there may be more than one 'acceptable solution'. However an analysis of the possibilities that both players have for winning makes it possible to share out the bets based on probability.

In fact, a maximum of 2 more rounds must be played to complete the game. There are four possible (and equally likely) results for these two rounds: (P, P), (P, R), (R, P), (R, R), where P indicates that Penny has won and R indicates that Rohan has won. In three of these scenarios, Penny, who only needs one more point, will win; Rohan will only win in one of the scenarios (the last). As such, the money from the bets should be distributed in the ratio 3:1, or rather 3/4 to Penny and 1/4 to Rohan.

Another of the problems from the correspondence between Pascal and Fermat relates to a betting game, specifically the decision of how to distribute the winnings of a betting game among the players if play is interrupted at a given moment. This problem, known as the *problem of points*, had first been tackled by Cardano, who had provided a solution based on the points already won by each player and not on the probabilities that each would win in the event that the game continued until the end.

A matter of counting. Does order matter?

It should be remembered that the probability of an event occurring is obtained by the application of the following rule: P (event) = favourable cases / possible cases, that is to say determining the number of times an event can happen and dividing this by the total number of possible cases. On some occasions, the calculation is extremely simple. By means of example, what is the probability of obtaining an even number when throwing a die? There are three favourable cases (obtaining 2, 4 or 6) out of a total of 6 possible cases; thus, p (even) = 3/6 = 0.5. Given that the total number of cases is extremely small, the favourable cases can be counted by simply listing all the cases. However, there are also occasions when the counting of favourable and/or possible cases can be significantly more complicated, and as such it is important to correctly identify the situation and have methods for calculating the number of cases. This means that an extremely important part of the analysis of a game of chance, or any random situation in general that has a certain complexity, consists of listing all the cases to be counted correctly.

We shall now analyse some situations that allow us to see the different ways of counting.

Situation 1

In an athletics race with 12 runners, how many different podium line ups (of the first three positions) are possible?

Any of the 12 runners could come first. For each of these, 11 athletes may come second and for each of these, there are 10 athletes who can come third. Thus the number of different podium line ups will be: $12 \cdot 11 \cdot 10 = 1{,}320$.

What this problem has done is to determine the number of groups of 3 athletes from a possible total of 12, taking into account the order in which they finish. Here

the list 1, 2, 3 is not the same as the list 2, 3, 1, even though the same athletes step up to the podium in both cases. In the first, athlete number 1 has won the race (number two has come second and number three third), while in the second, the winner is athlete number 2 (with 3 second and 1 third).

The previous example is known as the number of variations of 3 within 12 elements: $V_{12,3}$ and, as we have seen, is calculated by the product of $12 \cdot 11 \cdot 10$. In general, to calculate the number of variations of n within m elements (where $n < m$) the following formula should be used:

$$V_{m,n} = m \cdot (m-1) \cdot (m-2) \cdot \ldots \cdot (m-n+1).$$

Situation 2

In bridge, how many different ways can a hand of 13 cards be ordered?

If there are 13 cards and we wish to count the numbers of ordering them in all possible manners, there are 13 possibilities for the first card, 12 for the second, 11 for the third, and so on until the last card, which only has one possibility – the only card left over. Thus the total number of orderings will be:

$$13 \cdot 12 \cdot 11 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = 13! = 6{,}227{,}020{,}800.$$

The above operation is known as the number of permutations of 13 elements, and the result can also be written using what is referred to as factorial notation, placing an exclamation mark after the first number, in this case, 13!. In general $n!$ corresponds to the result of multiplying n by all previous numbers all the way down to 1. A table like the one below, with the first 12 factorials, gives an idea of the development of this series of numbers:

1!	1	7!	1 · 2 · 3 · 4 · 5 · 6 · 7 = 5,040
2!	1 · 2 = 2	8!	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 = 40,320
3!	1 · 2 · 3 = 6	9!	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9 = 362,880
4!	1 · 2 · 3 · 4 = 24	10!	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9 · 10 = 3,628,800
5!	1 · 2 · 3 · 4 · 5 = 120	11!	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9 · 10 · 11 = 39,916,800
6!	1 · 2 · 3 · 4 · 5 · 6 = 720	12!	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9 · 10 · 11 · 12 = 479,001,600

Counting is fundamental in many card games; Card Players, *a canvas by Lucas van Leyden (1520).*

Situation 3

In the game of bridge, how many different possible hands can be dealt from a deck of 52 cards?

In this instance, what we need to calculate is the number of different groups of 13 cards that can be formed from a total of 52, taking into account that once 13 cards have been selected, their order is irrelevant. One possible way of calculating the different possible hands is to consider that if the order were relevant, the total number would be:

$$52 \cdot 51 \cdot 50 \cdot \ldots \text{ (13 terms) } \ldots \cdot 42 \cdot 41 \cdot 40 = 3.95424 \cdot 10^{21}$$

However, since order does not matter we should note that each group of 13 cards has been counted 13! times (the number of permutations of 13), meaning that the number of different hands in the game of bridge is:

$$(52 \cdot 51 \cdot \ldots \cdot 41 \cdot 40) \;/\; 13! = 52! \;/\; (39!\ 13!) = 635{,}013{,}559{,}600$$

Note that the numbers we obtained are enormous. In the first example, when order is being taken into account, this gives a 22-digit number. In the second (without taking order into account), it gives a 12-digit number. It is possible to compare these numbers with the age of the universe: $1.5 \cdot 10^{10}$ years, which, expressed in seconds would be approximately $4.7 \cdot 10^{17}$ seconds. That is to say the first of the numbers ($3.9 \cdot 10^{21}$) is more than 8,000 times the number of seconds which have elapsed since the Big Bang, while the second number ($6.3 \cdot 10^{11}$) is 42 times the number of years that have passed since the start of the universe.

The situation above is known as the number of combinations of 13 elements from a set of 52: $C_{52,3}$ and, as we have seen, is calculated by carrying out the operation 52! / (39! · 13!). To calculate the number of combinations of n within m elements (again where $n < m$), the following calculation must be performed:

$$C_{m,n} = m! \,/\, (m - n)! \cdot n!$$

Situation 4

When the final of a football tournament ends in a draw, it goes to a penalty shoot out, generally with 5 kicks, each of which must be taken by a different player. How many lists of 5 players can be made from the team of 11 that finished the match in order to determine who will take the penalties?

In such a case, it is not clear if the order is relevant or not, and both interpretations are permitted, and there are two possible interpretations of the above problem.

a) Forming groups of 5 players, such that two different groups have at least one different player. In this case, the number of combinations of 5 within 11 elements is calculated by: 11! / (5! · 6!) = 462.
b) However, those familiar with the game will know that each team must submit an ordered list to the referee stating which player will take each of the 5 penalties in order. Thus, two lists with the same players in a different order will be considered as different. Therefore the number of variations of combining 4 elements from 11 must be calculated: 11! / 6! = 55,440.

Lottery numbers and other false intuitions

Let's listen in to a fictional conversation:

"Give me a number for the lottery"

"Here, take 00010."

"I don't want that one, it's really low and never comes up."

"If you want I'll give you 00001 as well; you can have two for the price of one."

"I don't want that one either, it comes up even less often."

"Okay, well I've got 74283."

"I like that one! I'll take it and thanks for letting me change."

We all have our own specific ideas of what chance is and what the rules of the game are. However, when faced with apparently simple probability problems, doubt often creeps in, in a way that is much more pronounced than with other types of mathematical problems and games. Thus when trying to mathematically model chance using probability, it is necessary to analyse each of the situations in detail. Perhaps the conversation above is an exaggeration, but it attempts to show to what extent the most basic rules of probability are far removed from many everyday situations, particularly when it comes to games of chance. On the other hand, the way people bet on sport and other events is another example of how little attention the general public give to the calculation of probabilities. Even when it shows that there is an extremely high probability of losing many games, and that even players who bet each week are highly unlikely to ever win, people continue to do so with the well worn argument that one day it will be their 'turn'. However, the same argument is not used when we go out for a drive and consider the chance of being involved in an accident.

The whims of probability

We will now consider some curious examples of the probabilities of winning a game or making a fair draw, which will hopefully make us think about and, on occasion, doubt our intuitions. All these games and problems will show that, in general, our knowledge of chance is less well grounded than we think, to the extent that our instincts often lead us to believe the opposite of what is actually likely to happen.

Playing bowls

Two friends, John and Charles, both bowls enthusiasts, are entertaining themselves with the following game: John has two bowls and Charles just one. They position the jack and bowl towards it. If both players are of the same ability, what is the probability that one of John's bowls stops closest to the jack?

The answer would appear to be 2/3 since Charles's only bowl can be first, second or third, and in the last two situations, John's bowls would be closest. However, thinking about the problem in a different way gives four possible cases: John's two bowls could stop in front of Charles's bowl, the two could stop behind, or one could stop in front and the other behind, or vice versa. Under these circumstances, Charles will still only be the winner in one scenario, but the probability of John's bowls being closest has gone up to 3/4. But which of the two arguments is incorrect? And why?

The first method of reasoning is the correct one. If the bowls are not marked, there are three possible scenarios, whereas if they are marked, the number of results is six and in four of these, one of John's bowls will be closest to the jack. The second argument is incorrect because only one of the possible general results was divided in two – when Charles's bowl is in the middle – to take into account the specific positions of John's bowls. If we do that with one general scenario, we must do it with the other two – when Charles's bowl is first and when it is last.

A standard die

Brenda and Roger have a standard die, that is to say its six faces are marked with the numbers from 1 to 6. Brenda rolls the die first, then Roger. What is the probability that the number obtained by Brenda will be higher than the one obtained by Roger?

It is clear that the probabilities of the numbers will be 1/6 (Roger has a one in six chance of throwing the same number as Brenda). Thus, the probability that they will be different will be 5/6. The probability that Brenda's number will be higher is half of this, 5/12.

What is the probability of winning?

Consider three different coloured dice. A red one with the numbers 2, 4 and 9 on its faces, each number appearing twice; a blue one with the numbers 3, 5 and 7, also

A 1st-century fresco from Pompeii depicts dice players.

appearing twice, and a white one with the numbers 1, 6 and 8, repeated twice as on all the other dice. The game is played by two players who take turns to select a die and roll it, with the person who obtains the highest score winning. By letting one's opponent choose first, it is always possible to choose a die that gives the highest chance of winning. How is this possible? Which die should you choose?

In spite of the fact that the numbers of all the dice add up to the same total, there is something surprising at work here. The blue die beats the red one, the white beats the blue and the red beats the white. In all the pairings, given nine throws, on average five of these will be won by the first die and four by the second. This means that the probabilities, which can be easily calculated by analysing all possible cases for each pair of dice, are 5/9 for winning with one of the die and 4/9 for winning with the other. Thus, provided they always make the correct choice, the second player to choose a die will always have a greater chance of winning.

A disputed draw

A teacher decides to raffle a prize among the 30 students in her class. One suggests taking 30 pieces of paper, marking each of them, folding and mixing them, and distributing one to each student. The teacher suggests a simpler and quicker method: "I'll think of a number between 1 and 30 and write it down on a piece of paper, then proceeding in the order in which you are seated, each student will give me a different number until they guess the number I'm thinking of." One student at the back of the class voices his disapproval of this method, arguing that he will have a very small number of possibilities to choose from, less than the first students, and in all likelihood he wouldn't even have the opportunity to pick a number because one of his classmates would guess correctly first. Is the student right, or has the teacher suggested a fair way of making the draw?

The teacher's reasoning is completely balanced and each student will have the same probability of getting the number right, 1/30 for each. In fact it is clear that for the first student, the probability is 1/30 since there are 30 numbers to choose from. The probability for the second will be: $29/30 \cdot 1/29 = 1/30$, or rather the probability of the first getting it wrong (29/30) and them getting it right (1/29); for the third it will be $29/30 \cdot 28/29 \cdot 1/28 = 1/30$, and so on until the final student. In addition to this, note that the probability for the first student is 1/30. If the probability were to decrease for the others, the sum of the probabilities would not be 1; this would be impossible since going through all the numbers, one will always be correct.

An uninteresting bet

A roulette player always bets on an odd or even number (if they win, they double their bet and if they lose, they lose it). He decides to play in the following way: beginning with a certain amount of cash, each time he will bet 1/10 of the total at that moment. If he starts with £100 and makes ten continuous bets, winning 5 and losing 5, will he have more, less or the same amount of money than when he started? The problem can be generalised, assuming that the player starts with any value, m and that each time they bet $1/n$ of the amount they have at the time of making each bet.

Although it may appear that after playing ten games, winning five and losing the others, our player will have the same amount of money as at the start, he will in fact have less. When winning a game, the amount increases by 1/10, equivalent to multiplying the quantity by 1.1, whereas when their bet is wrong, they lose 1/10,

An illustration from the 18th century shows caricatures of players of "even-odd", a predecessor to Roulette.

equivalent to multiplying the quantity by 0.9. In this manner, with five correct bets and five incorrect ones (regardless of the order in which they have occurred), they will have: $100 \cdot (1.1)^5 \cdot (0.9)^5 = 100 \cdot 1.61051 \cdot 0.59049 = 100 \cdot 0.95099 \approx$ £95.099, having lost almost £5. The argument can be generalised and the fact that the end result will always be less than the number at the start is based on the fact that $(1 + 1/n) \cdot (1 - 1/n) = 1 - 1/n^2$, which is less than 1, meaning that the initial number, being multiplied by a number that is less than one, will always decrease.

Shared birthdays

Here is another basic probability problem with a surprising result: what is the probability that in a group of 25 people there are at least two who celebrate their birthday on the same day? Taking into account that there are 365 days in the year (not counting leap years) and that there are only 25 people in the group, we often instinctively

believe that the probability we are looking for will be very low, at least less than 0.5, (more unlikely than likely), whereas a calculation based on the principles of probability shows that it is in fact greater than 0.5 (more likely than not).

Indeed, given that it is possible that two or more people were born on the same day, all we need do is calculate the probability that everyone in the group was born on a different day. To do this, let us consider the 25 people in order. The first person could have been born on any of the 365 days, the second on any of the 364 remaining days, the third on one of the 363 remaining days, and so on. Hence the probability that the 25 people were born on different days will be:

$$p \text{ (different day)} = 365/365 \cdot 364/365 \cdot 363/365 \cdot \ldots \cdot 341/365 =$$
$$= 365! \,/\, (340! \cdot 365^{25}) \approx 0.4313.$$

Based on this result, we can deduce that the probability that at least one of the two people share the same birthday is: $1 - 0.4313 = 0.5687 > 1/2$. In fact, all we need is a group of 23 people for the probability to be greater than 1/2.

Chance has no memory

One of the aspects in which our intuition often fails us is in determining the likelihood of independent events. Assume we are watching a game of roulette and there has been a run of 10 even numbers. We have to decide whether to bet even or odd on the next game. Which would be best? Without a shadow of doubt, with the most rudimentary knowledge of probability we can say that it does not matter, since the probabilities of an even and odd number coming up are the same. However, this idea, which is often summarised by the saying "it makes no odds", is not always so easily identified, as we shall see in our analysis of the following situations.

Tossing a coin

A mathematics teacher suggests his students toss a coin a large number of times, for example 150 tosses, and write down the results, writing 1 each time the coin lands heads up and 0 each time it lands tails up. These are the results provided by two of the students:

Rasha: 01011001100101011011010001110001101101010110010001
0101001110011010110010110010110010010111011001101 1
01010010110010101100010011010110011101110101100011.

Luke: 10011101111010011100100111001000111011111101010101
11100001010001010010000010001100010100000000011001
00001001111100001101010010010011111101001100011010.

The teacher looks at the results and decides something is wrong. While it is clear that one of the students has carried out the experiment correctly, the other thinks that there is no need to toss the coin and they can get away with writing a line of ones and zeros at random. Unfortunately their idea of chance is still inaccurate and the teacher quickly discovers that one of the students has cheated. Which of them did not toss the coin?

The regular distribution of Rasha's ones and zeros leads the teacher to suspect it is she who has cheated in the experiment. In fact, if Rasha and Luke's distributions are compared, it can be seen on the one hand that the numbers of ones and zeros in each is similar and 'reasonable' (78 and 72 in the case of the former, and 70 and 80 for the latter), however in Rasha's case, the sequences of ones and zeros are still very small (at most three), while in Luke's series, there are groups of four, five and up to nine similar digits. The teacher has discovered her prime suspect.

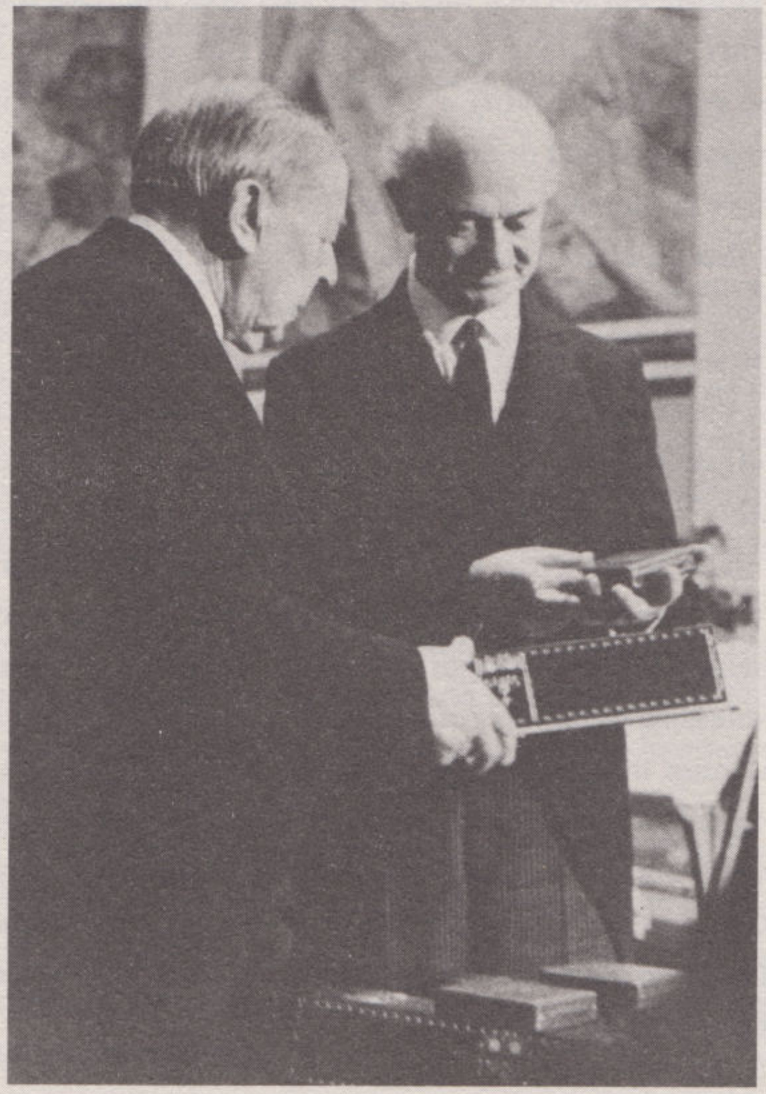
Analysing the previous information in terms of conditional probability and taking into account that each toss is independent of the previous ones, it is clear that after a one, there must be a series of 'reasonably' distributed ones and zeros. In fact, in Rasha's distribution, after a one there are 47 ones and 30 zeros, after two ones there are only 5 ones and 18 zeros and after 5 sequences of three ones there is always a zero. It can be observed that this clear deviation also occurs when taking the sequences of zeros in Rasha's distribution, although it does not occur in Luke's results (e.g. after two ones there are 18 ones and 14 zeros, and after three ones there are 9 ones and 9 zeros). As such, Rasha's version of chance, which lacks irregularities, was what alerted the teacher to her trick.

However, it is in the following situation where the decision regarding the influence of information on the modification (or not) of probabilities reaches its most interesting point. The game described below, an adaptation of the classic problem of 'prisoners', shows the difficulty of arguing how a given piece of information alters probability.

THE MODESTY OF A TWO-TIMES NOBEL PRIZE WINNER

When the chemist Linus Pauling (1901–1994) received his second Nobel Prize (the first was the 1954 Nobel Prize for chemistry, for his work on quantum chemistry, while the second was the 1962 Nobel Peace prize for his campaign against nuclear testing), he remarked, clearly joking, that while obtaining the first prize was extremely difficult, since the probability was around one in six billion (the population of the world), the second had much less merit, since the probability was one in only a few hundred (the number of people alive who have previously received the prize). What is the problem with this entertaining but false reasoning?

In order to be able to say that the probability of receiving a second Nobel Prize only depends on the number of people who have already received a first, we must know that the committee has decided to award the prize to a person who has already been awarded a Nobel Prize. Without this information, obtaining the second prize, at least in probabilistic terms, is just as hard as obtaining the first, since it assumes that in its selection process the committee does not take into account whether the candidates have previously received other prizes. In this case the idea of viewing the achievement of a Nobel Prize in terms of probability is in itself clearly a joke, since it is evidently not only a matter of luck but, primarily, one of merit.



Linus Pauling (right) being presented with the Nobel Peace Prize in 1962.

Gameshow

One of the challenges in a television game show involves trying to find a prize that is hidden behind a door. Three doors are presented to the contestant, and they are asked to choose one (without opening it). The presenter (who knows which door conceals the prize) opens one of the doors that has not been selected by the contestant – and which does not conceal the prize. The player is then ask if they wish to swap the door they had initially selected for the other one which is closed. If they agree to the swap, will they have more chance of winning the prize?

This is a new version of a famous and controversial problem about probability, in which we must consider how the probability of each of the doors has changed. When the contestant selects one of the three doors, the probability of finding the prize is 1/3. When the presenter selects one of the other doors (one which does not have the prize) and opens it, the probability of the first is unchanged, since it is already known that one of the two does not have the prize. However the probability of the other unselected door (the one which is still closed) does change and goes from having a probability of 1/3 to 2/3 (the probabilities of the two remaining closed doors must add up to 1). Thus the contestant should always swap doors to increase their probability of winning to 2/3. The controversy generated by this problem is to be found in the fact that, as suggested, the probability of the door initially selected by the contestant does not change. It would be different if, instead of the presenter opening one of the doors that does not conceal the prize, the contestant selected one of the two remaining doors left after their first choice, and asks if the prize is there – the presenter replies yes or no. In this case, the probability of the first selected door conceals the prize increases from 1/3 to 1/2.

The above game gives rise to an interesting generalisation. Assume there are n doors with a prize behind one of them. The contestant selects a door (without opening it), the presenter opens one of the others without the prize and allows the contestant to switch doors. The presenter then opens another door (from among the ones that are still closed (excluding the last one selected by the contestant), which does not conceal the prize, and gives the contestant another chance to switch doors. The game continues until there are only two doors closed and the player chooses for the last time. How must the contestant act throughout the game in order to maximise the probability of obtaining the prize? What is the probability of winning in this game?

Starting from the fact that when the presenter opens a door, the probabilities of all the closed doors are modified, except for the one selected by the contestant. This means that the strategy that maximises the probability of winning consists in not switching doors until only two are left closed, at which point the contestant should switch doors and will now have a probability of $(n - 1) / n$ of finding the prize. Furthermore, on their first choice, the probability of finding the prize is $1/n$ (remember that there are n doors). If they do not switch until only two doors are left closed, the door that was initially selected will continue to have probability $1/n$, meaning that the other closed door will have probability $(n - 1) / n$, the highest possible. If, on the other hand, they switch doors at an intermediate step, although it is now more complicated to determine the probabilities (it depends on the number of changes and at what point they are made), it is certain that all exceed $1/n$ (the probability of each has increased at least once). That means that when there are two closed doors, none will have probability $(n - 1) / n$. If we wish to study this game in greater depth, we could consider how the probabilities change in line with the various strategies. The results are complex but very interesting.

Mathematics and expectations

One of the most important concepts for making decisions in games of chance is what is referred to as the *mathematical expectation*. Let us consider a few examples before defining the term fully. Suppose we are offered the chance of playing the following game: two coins are tossed in the air and if the result is two heads, we win £4, if the result is two tails, we win £1, and if the result is a head and a tail, we lose £3. Should we be interested in playing this game? How much do we expect to win – or lose?

There are four possible results when tossing two coins: two heads ($p = 1/4$), two tails ($p = 1/4$), and a head and a tail ($p = 1/4$), tail and head ($p = 1/4$). Thus on average, four tosses will give the following: two heads; two tails and two sets of one head and one tail. This means that on average, the winnings will be $1 \cdot$ £4 $+ 1 \cdot$ £1$+ 2 \cdot$ (-£3)= -£1. This suggests that we should not play and that if we do so, we will lose on average £1 for each four rounds, or rather, 25 pence per game. The same result can be obtained by multiplying the probability of each of the possible cases by the winnings for corresponding case (or losses, where they are negative) and adding the results together.

In this case, we have:

$$1/4 \cdot £4 + 1/4 \cdot £1 + 1/2 \cdot (-£3) = -£0.25$$

Let us consider a second example. In a betting game that consists of throwing a die, the bank pays 6 chips if a 6 comes up, 4 chips if an odd number comes up, and nothing in the other cases. How much should we bet each round to ensure the game is balanced?

Taking into account that p (6) = 1/6 and p (odd) = 1/2, it is expected that the following will be won in each game: $1/6 \cdot 6 + 1/2 \cdot 4 + 1/3 \cdot 0 = 3$ chips. Thus the game is balanced (is neither in favour or against the player) with a bet of 3 chips.

These examples allow us to introduce the idea of the mathematical expectation and balanced betting games, which can now be defined in a more general manner. Let $E_1, E_2, E_3, .., E_n$ be events that cannot occur simultaneously that arise in a game of chance, each with a probability of $p_1, p_2, p_3 ..., p_n$ (where $p_1 + p_2 + p_3 + \ldots + p_n = 1$), and with respective payoffs of $r_1, r_2, r_{3...}, r_n$, the expected winnings or *mathematical expectation* (X) of a game (or a random experiment) in which the results must be one of the events $E_1, E_2, E_3 \ldots, E_n$, is defined as:

$$X = p_1 \cdot r_1 + p_2 \cdot r_2 + p_3 \cdot r_3 + \ldots + p_n \cdot r_n.$$

Based on this definition, a betting game is said to be fair (or balanced) if the mathematical expectation (the average winning events per game) is the same as the bet that must be paid. It is also said that the overall mathematical expectation (the expected winnings minus the bet) is 0.

Let us now consider a new application of the mathematical expectation in order to decide if a game of chance is balanced or not.

A betting game with three dice

A game of chance consists of the following: a player bets £1 on a number from 1 to 6 – let's say 3. Three normal dice are rolled and if a three comes up, they win £1, if two threes come up they win £2 and if three threes come up, they win £3. In each case, they also recover their £1 bet. If none of the three dice is a three, they lose their bet. Is this game balanced, in favour of the player or the bank?

Although at first sight it may seem like the game is in favour of the player, in fact it is not. It could appear tempting to bet if we argued along the following lines. Since there are three dice and the probability of obtaining the winning number is 1/6 for each die, there will be at least a 1/2 chance of winning. However, there is also the chance of getting two or even three numbers, meaning the game is in favour of the player.

However the above reasoning is incorrect. There are in fact 216 possibilities ($6 \cdot 6 \cdot 6$). The triple only comes up in one case ($p = 1/216$); the double comes up in 15 cases ($p = 15/216$) and in 75 cases the player doubles their bet ($p = 75/216$). Thus the player loses their bet in 125 cases (216 - 1 - 15 - 75).

There are more cases in which they lose (125) than in which they win (91). Calculating the mathematical expectation associated with a £1 bet gives:

$$3 \cdot 1/216 + 2 \cdot 15/216 + 1 \cdot 75/216 - 1 \cdot 125/216 = 108 / 216 - 125 / 216 = -17/216 = -0.0787\ldots$$

Thus the game is in favour of the bank, which can expect to win almost 8 cents for each Euro bet.

Although this provides an example of the idea of the mathematical expectation in games of chance, the concept is also applicable to a wide range of random situations, which in many cases have little or nothing to do with games of chance, such as the following.

Early registration

Let's say that next July there is a conference that you are interested in attending although you are not sure if you will be able to make it due to your workload and other commitments.

When paid before 1 March, the registration fee is £150 (without the chance of a refund if you are unable to attend), whereas if paid after this date, the fee is £200 (this can even be paid upon arrival at the conference).

On the 28th February you estimate the likelihood of attending the conference (let's call this probability p). What can the value of p tell you about whether to pay in advance or wait until you arrive at the conference?

If you pay in advance, the expectation is to lose £150 (regardless of whether you attend or not, since there is no refund available).

If you pay upon arrival, the expectation is to lose £200 $\cdot p + (1 - p) \cdot 0 = 200 \cdot p$ (only paid if you attend the conference).

The two expectations are equal if $p = 150/200 = 0.75$.

Thus if $p > 0.75$ it is better to pay for early registration and if $p < 0.75$ it is best not to pay until arriving at the conference. Where $p = 0.75$ the result is the same either way.

Can the bank be beaten? Probability and repeated events

As we have seen in the previous section, the mathematical expectation gives an idea as to whether a betting game is balanced or not. In the former case, after a large number of games, the player expects to make neither a loss nor a gain, whereas in the latter, we have shown how to determine the quantity that they expect to win (or lose). However there have been, and still are, players who, after repeatedly making bets on a balanced game, or one with a slightly negative expectation, have managed to make large winnings. Let us make use of mathematics to better understand the relationship between repeated games (or trials) in a game of chance (or experiment), in order to find ways of determining the probability of 'exceeding expectations'.

Let us begin by analysing a problem that comes up in the game of roulette (with 37 numbers, from 1 to 36, plus 0). What is the probability of obtaining three zeros in 10 games?

The probability of obtaining 3 zeros in a given position will be: $(1/37)^3 \cdot (36/37)^7 = 0.00016$. The total probability will be as above because of the number of positions that can be occupied by the three zeros: $C_{10,3} = 120$, or rather:

$$p \text{ (3 zeros in 10 games)} = 120 \cdot 0.00016 = 0.0192,$$
$$\text{approximately a 1 in 50 chance.}$$

The previous example can be generalised in the following way to obtain an important result for the analysis of games of chance. If in a game of chance (or randomised experiment), n games take place (n independent trials), and it is known that the probability that a certain event (success) related to the game will be repeated is p, then:

$$p \text{ (} r \text{ successes in } n \text{ experiments)} = C_{n,r} \cdot p^r \cdot q^{(n-r)}, \text{ where: } q = 1 - p, r \leq n.$$

The probability distribution when taking r of the different values from 1 to n is known as the *binomial distribution*. In order to be able to apply this distribution, the experiments must be independent and the probability of the event must be constant for the successive experiments.

We can use this probability distribution to find the probability of getting r heads when tossing a coin n times, with $r = 1, 2\ldots, n$. In this case, p (one head) = 1/2, and thus $q = 1/2$, where we will always have $p^r \cdot q^{8-r} = (1/2)^r \cdot (1/2)^{8-r} = (1/2)^8 = 1/256$. Multiplying this value by the successive combinations ($C_{8,r}$) for different values of r gives:

Number of heads	Number of ways of obtaining the number of heads	Probability of obtaining the number of heads
0	$C_{8,0} = 1$	$1 \cdot 1/256 = 1/256$
1	$C_{8,1} = 8$	$8 \cdot 1/256 = 8/256$
2	$C_{8,2} = 28$	$28 \cdot 1/256 = 28/256$
3	$C_{8,3} = 56$	$56 \cdot 1/256 = 56/256$
4	$C_{8,4} = 70$	$70 \cdot 1/256 = 70/256$
5	$C_{8,5} = 56$	$56 \cdot 1/256 = 56/256$
6	$C_{8,6} = 28$	$28 \cdot 1/256 = 28/256$
7	$C_{8,7} = 8$	$8 \cdot 1/256 = 8/256$
8	$C_{8,8} = 1$	$1 \cdot 1/256 = 1/256$

The symmetry of the probability distribution seen in the above table is because the probability of obtaining one head when tossing is 1/2. The reader will most likely have observed that the numbers of the sequence (1, 8, 28, 56, 70, 56, 28, 8, 1) in the previous table, which add up to 256 (2^8), correspond to the numbers from a row of Pascal's triangle. In fact, the binomial distribution is related to the coefficients of the binomial series and, in this specific case, corresponds to the successive coefficients of $(a + b)^8$.

Chapter 4

Game Theory

Nine tenths of mathematics, outside of what has been required for practical needs comes from the solutions to riddles
Jean Dieudonné

Game theory is a branch of mathematics that mainly deals with decision making. It is applied to all sorts of situations in which there is conflict, in which the participants must make decisions that are in their best interests, without knowledge of those made by their opponents. The formulation of the theory is based on abstract games, hence its name, although its interests do not really lie in games. Instead the theory applies the ideas of games to analyse and solve all sorts of problems.

This chapter is focused on competitive two-player zero-sum games. The term 'zero-sum' means that at all times the gains of one player are equivalent to the losses of the other, that is to say, there is only one winner and that this winner takes all. It is assumed that each player always tries to make the move that benefits them the most, or rather the one that results in the greatest gains. In other words, the players are never satisfied with less than the totality of the gains.

The principles of game theory

As an introduction to game theory, let us now consider three games that serve to distinguish varying levels of difficulty, as well as certain key concepts that will be used throughout this and the following chapter. The reader should understand that although the theory uses the terminology of games, and as such, talks of games, players, rounds, strategies, balanced games, the value of a game, etc., each of the scenarios presented here do not really correspond to a game in the sense that we have used the term in the previous chapters. It is better to imagine a situation or conflict, initially between two people (or groups of people) in which there are rules determining the

PRECURSORS TO GAME THEORY

Portrait of G.W. Leibniz, a German philosopher who also made many contributions to mathematics.

In the 17th century, scientists such as Christiaan Huygens (1629–1695) and G.W. Leibniz (1646–1716) were already proposing the creation of a discipline that used scientific methods to study human behaviour and conflicts, although they did not achieve any significant results. The 18th century saw very little work related to the analysis of games for this purpose. However this included a letter written by James Waldegrave in 1713 providing a solution to a card game (*Le Her*) that was restricted to two people and applied a method similar to that now known as a mixed strategy to give a minimax type of solution. However, there is no type of theorisation or generalisation to allow the method to be applied to other cases.

In the 19th century, a number of economists developed simple mathematical models for the analysis of basic competitive situations. These included the work of Antoine Augustin Cournot, *Recherches sur les Principes Mathématiques de la Théorie des Richesses* (1838), which considers a duopoly and provides a solution that can be thought of as a specific case of Nash's equilibrium. However, game theory as a well-founded branch of mathematics is essentially a product of developments during the 20th century.

possible moves, which each of the players makes simultaneously – not alternately as in chapter 2 – meaning that they do not know the move made by their opponent, one player makes a gain and the other a loss. Thus from here onwards, we shall talk of *games* to refer to situations; of *players*, of which there are at least two involved in the scenario; of *strategies*, in which each player will make decisions that equate to moves; and of *gains*, or rather, the value won or lost as a result of each decision.

In order to initiate ourselves in the fundamentals of game theory, let us begin with the following case, which is extremely simple and of no interest as a game. Two people, A and B, must simultaneously write either the number one or two. Player B must pay player A the value in pounds of the sum of the two numbers written by each. This is evidently not a balanced game since A will always win. However, we can ask how each player must play in line with their interests. Let us consider the game as a matrix, known as ae *payoff matrix*, with the following possible results:

	B writes 1	B writes 2
A writes 1	2	3
A writes 2	3	4

The numbers in the matrix indicate the value in pounds that B must pay A, depending on the strategy chosen by each player (the two possibilities for each player give the four results of the matrix). Given the simplicity of the game, it becomes evident that if each player plays in line with their interests, A will write two, while B will write one, meaning that A's gain will be £3.

Let us consider these moves in greater detail in order to see how each player proceeds: given that A does not know B's move, they must assume that B will play to minimise the amount they must pay, such that if A writes one, they will gain a minimum of £2, and if they write a two, they will gain a minimum of £3. It is said that 3 (the number in the bottom left square of the matrix) is the *maximin* (maximum of the minimums). Similarly, B assumes that A will play to obtain the maximum benefit, meaning that if B writes one, they will lose a maximum of £3 and if they write two they will lose a maximum of £4. It is said that 3 is the *minimax* (the minimum of the maximums). When the maximin and minimax are in the same square for a given game, as is the case here, it is said that the game is *strictly determined* and that it has a *saddle point* (imagine a riding saddle and two perpendicular curves, one with a minimum and the other with a maximum and a point where the minimum of one coincides with the maximum of the other).

The value corresponding to this saddle point, in this case £3, is the *value of the game*, which is always obtained when each player follows their optimal strategy. If one of the two players makes a different move (applies another strategy), their opponent will be able to increase the value of the game, gaining more or losing less, depending on whether they are A or B. It is also said that this is a *determined game* for which there is a *pure strategy*.

Let us now consider another game with the same conditions as above with respect to the moves that can be made by both players, but with a different payoff matrix determined by the criterion of equality: if both have written the same number, A wins £1, whereas if they are different, B wins £1.

	B writes 1	B writes 2
A writes 1	1	–1
A writes 2	–1	1

Now A's maximin is –1 (both minimums are –1), whereas B's minimax is 1 (both maximums are 1); the difference means that this game does not have a saddle point and as such there is no pure strategy for playing it. If A adopts a strategy (such as always writing a one) and this is identified by B, B will systematically write a two and will always win £1. Given the simplicity of the game and its symmetry, the optimal strategy must be one that contains a proportional number of ones and twos such that the opponent is unable to identify a pattern. As such, the optimal strategy consists of playing randomly, for example tossing a coin in the air and writing one if it lands heads and two for tails. Under these circumstances, it is impossible to speak of pure strategies, since the required element of chance means that the game cannot be determined in advance. When the optimal strategy requires the use of chance and must be kept secret, we can talk of 'mixed strategies'.

These two examples can be thought of as extremes. In the first, the game is determined by the selection of a pure strategy, since the best strategy for each player leads to a consistent result referred to as the 'value of the game'. However in the second, a predetermined strategy for playing does not necessarily lead to the best results and the only way of guaranteeing this is through the application of a random strategy, referred to as a 'mixed strategy'.

Let us now consider another game, similar to the previous ones, but with a more complex analysis of the optimal strategies for each player. Like in the previous games, each player can write two numbers. Player A can write either 1 or 8 and player B can write either 7 or 2. If the numbers written by both players have the same parity (both even or both odd), A will win their total value in pounds, whereas if one is even and the other odd, B will win, again with the value of their winnings determined by the number they have written.

The payoff matrix for this game is as follows:

	B writes 7	B writes 2
A writes 1	1	–2
A writes 8	–7	8

Remember that the numbers of this payoff matrix refer to the winnings of player A; as such, when B wins, a negative number is written to represent the loss to player A. A can win £1 or £8, whereas B wins £2 or £7. No saddle point exists; the maximin is -2 (- 2 > -7), whereas minimax is 1 (1 < 8). In fact, when in a 2 x 2 matrix the numbers of a diagonal are greater than the other two values, there is never a saddle point. That means that the game is not determined and there is no pure strategy. However, in contrast to what happened in the previous game, in which the best strategy for the two players was to opt for playing randomly in order to balance gains, in this case, B has a way of winning. This time the optimal strategy for each player, despite still being to some extent random, is not strictly so. Each player makes their decision in line with determined proportions. Also in this case, the solution to the game makes use of mixed strategies by each player. We shall return to the results of this game, including the determination of the optimal strategy for each player later on.

The reader will have observed that the various games have been presented using a matrix the rows of which list different strategies for the first player and the columns list the strategies for the second. This representation, known as the 'normal form' of a game, is the most common for two-player games in which the moves occur simultaneously, something that happens in the majority of situations analysed by game theory. There is also another representation, referred to as the 'extensive form' of a game, which involves representing all the moves in a tree diagram. This is best suited for games in which players make alternate moves. The majority of games in Chapter 2 are of this type.

THE BIRTH OF GAME THEORY

The French mathematician Émile Borel carried out a number of studies in the field of probability theory.

Moving into the 20th century, there were already moves to formulate a theoretical framework which, by the middle of the century, would form the basis of what is now known as game theory. The first general theorem to be proved was in the work of Ernst Zermelo (1871-1956), finally formulated in 1912. The theorem states that for any finite complete information game (such as draughts or chess) there is an optimal solution based on pure strategies, that is to say without the requirement to introduce a random element. However the theorem only proves the existence of such a solution and says little or nothing about how to find such strategies.

In 1920 the great mathematician Émile Borel became interested in an emerging theory and introduced the idea of a mixed strategy (one which includes random elements). Soon after, John von Neumann began his work, formulating and proving the minimax theorem in 1928, which soon became a key element in the development of the theory. The theory states that in a finite game with two players, A and B, there is an average value, which represents the amount A can win from B if both play fairly, that is to say, trying to obtain the greatest gains or least losses.

When is equilibrium reached?

The games analysed in the previous section are simple for a number of reasons. There are two participants (two-player games) and each has only two possible moves (the payoff matrix is always 2 x 2). Furthermore, they are zero-sum games since the sum of the gains of the players always gives zero (a loss counts as a negative gain). For any given round, the strategies are reduced to one of two possible moves. In line with the conditions of the game, it may be that each player opts for a determined strategy (the optimal strategy for each player), by which the game is determined, alongside the result corresponding to the value of the game (as in the first example

JOHN VON NEUMANN (1903-1957)

John von Neumann was a versatile scientist and one of the most distinguished mathematicians of the 20th century. He began his work in the city where he was born, Budapest, studying mathematics before moving to Berlin to study physics and then Zurich, where he studied chemical engineering. In 1930 he moved to the United States. In Göttingen and under the tutelage of Hilbert, he worked on theoretical problems from pure mathematics, and also worked together with Heisenberg on the first formulations of quantum theory. He made significant contributions to a range of fields, including set theory, functional analysis, logic, probability, applied mathematical economics, quantum physics and meteorology.

His interests gradually began to shift from pure to applied mathematics, and to fields as diverse as atomic physics, the design of digital computers, cognitive psychology and economics. One of his principal contributions was in the area of applied mathematical economics, with the establishment of game theory in the book *Theory of Games and Economic Behaviour* (1944), published together with Oskar Morgenstern in Princeton. The work is regarded as the most important contribution to this branch of mathematics since it marks the consolidation of the theory which, a few years later, at the start of the 1950s would be applied to a large number of situations for the analysis of the real world.

John von Neumann (on the right) and Robert Oppenheimer, scientific director of the programme for the development of the first nuclear bomb, pose for a photograph in this image from 1952, in front of the fastest and most accurate computer constructed at the time.

of the previous section). We have seen that this is always the solution when the game has a saddle point, that is to say, when one of the values of the matrix is both the maximin (the maximum of the minimums for each row) and the minimax (the minimum of the maximums for each column). If this is not the case, it is no longer possible to use pure strategies and the players must instead turn to mixed strategies, which must be kept secret and selected by introducing an element of chance. In cases where the payment matrix is symmetrical, the strategy consists of making

the selection completely at random (as in example 2). Otherwise, even though a random strategy is still used, the selection of each of the possible moves must be weighted (as in example 3).

An abstract game with pure strategies

Let us now analyse the first type of game and see what happens when the matrix for the game is extended, so there are more than two possible moves for each player.

Let's start with the following two-player game: Player A selects a row (R1, R2, R3) and their opponent a column (C1, C2, C3) from the following matrix (the payoff matrix for the game), without either one being aware of the action of their opponent. The two selections determine a number in the matrix (intersection of the row and the column which have been selected) which indicates the value in pounds that the second player must pay to the first. How must each player play in order to maximise their gains or minimise their losses?

		Player B		
		C1	C2	C3
Player A	R1	5	–2	1
	R2	6	4	2
	R3	0	7	–1

Player A analyses their minimum payoffs according to the possible moves (–2 if they make move R1, 2 if they make move R2 and –1 if they make move R3); the best of the minimum payoffs (maximin) is 2. If the game is determined, they must select R2. Similarly, player B analyses the moves which give them the best losses, according to the possible moves (6 if they make move C1, 7 if they make move C2 and 2 if they make move C3.) The least of the maximum losses (minimax) is 2. If the game is determined, Player B will select move C3.

Given that the maximin and minimax coincide in this game and both result in a payoff of £2, it can be said that the game is determined, that its value is 2 and that it is solved by means of a pure strategy: A plays R2 and B plays C3. It can also be said that 2 is a saddle point (the maximum of the minimums coincides with the minimum of the maximums) or point of equilibrium.

This example can be generalised, keeping the same number of players but giving them n possible moves instead of 3, making the payoff matrix $n \times n$. Provided there is a saddle point, the game has a point of equilibrium associated with a pair of pure strategies (those which are optimal for each player). Such a game has a stable result because if one player were to change their strategy, they would be left worse off by carrying out this action and the position of their opponent would improve as a consequence.

ARE GAMES STABLE?

Let us now consider the analysis of the following matrices of two-player zero-sum games to determine if they are stable games by finding their saddle point or point of equilibrium.

		Player B		
		C1	C2	C3
Player A	R1	2	–5	–2
	R2	3	–1	–1
	R3	–3	4	–4

		Player B		
		C1	C2	C3
Player A	R1	–2	1	1
	R2	–3	0	2
	R3	–4	–6	4

		Player B			
		C1	C2	C3	C4
Player A	R1	–3	17	–5	21
	R2	7	9	5	7
	R3	3	–7	1	13
	R4	1	19	3	11

Elections and restaurants: applications of games with pure strategies

The method for resolving abstract games provided in the previous section can be used to analyse and solve a wide range of situations. Let us now consider two concrete examples.

Manifestos

Consider the following situation: one of the issues that has polarised opinion in a given country is the construction of a new bypass road around the capital. There are two possibilities: that the road passes round the North (N) or the South (S) of the city. Drawing up their manifesto, the country's main political parties, A and B, must decide if they are in favour of the construction of option N or S. They can also decide to avoid the issue and omit it from their manifesto. Both parties know that they will be supported by their members, regardless of their decision, although they know that the remainder of the population will select one or other of the options and in the event that both parties opt for the same, will abstain. Surveys have been carried out amongst the electorate. They are available to both parties and the results for party A are given in the following matrix.

		Manifesto for B		
		N	S	Omit
Manifesto for A	N	40%	45%	35%
	S	55%	50%	45%
	Omit	40%	50%	35%

Thus for example, if party A proposes the northern option and B the southern one, A will obtain 45% of the vote while if both avoid the issue, A will obtain 35% of the votes. Under these conditions which options should be taken by each of the political parties?

Based on the data in the above matrix, the decision is clear: party A will note that its best results always lie in choosing S. Likewise, party B will note that the worst

results for A (which are in its best interests) arise when B avoids the issue. This will be its option. As such, the situation has a point of equilibrium (A selects S and B decides to avoid the issue). The resulting payoff is 45% of the vote to A.

Let us now assume that the matrix is as follows:

		Manifesto for B		
		N	S	Omit
Manifesto for A	N	60%	55%	45%
	S	40%	20%	40%
	Omit	45%	20%	35%

A's decision remains clear. Under all circumstances, the best option is N, although B can no longer make their decision without taking A's action into account. The temptation to select S, in the hope it will leave A with just 20% is a bad move, since under these circumstances, if A chooses correctly, they will obtain 55% instead of 20%. This means that the best option for party B will be to avoid the issue and the result will be 45% of the vote to A.

Finally, let us assume that the matrix is as follows:

		Manifesto for B		
		N	S	Omit
Manifesto for A	N	35%	10%	60%
	S	45%	55%	50%
	Omit	40%	10%	65%

This time neither of the two participants can make an immediate decision since each depends on the actions of the other. This means they must consider what is their best option given any of the choices made by their opponent, or rather, which

is the least worst of the worst options. A will obtain a minimum of 10% if they select N, 45% if they select S and 10% if they avoid the issue, and hence should select S. Likewise, if B selects N, A can obtain a maximum of 45%, if they select S, A can obtain a maximum of 55% and if they avoid the issue, A can obtain a maximum of 65%, meaning that B must opt for N.

Under these circumstances, the best option for each party will give the same result: 45% of the votes for A, the saddle point of the situation.

The location of a restaurant

Two partners, Mary and George, want to open a restaurant and decide to do so at a crossroads, in the outskirts of a large city surrounded by mountains. They agree on all the conditions except one: Mary would prefer the location to be as low as possible, whereas George would prefer it to be as high as possible. In this respect their interests are completely opposed. In order to make their decision, they decide to organise a competitive game. They select three parallel motorways, M1, M2 and M3, which run from east to west and 3 roads, which are also in parallel, A1, A2 and A3, running from north to south. The points where the motorways cross the roads give 9 possible locations, whose heights in metres are provided in the following matrix.

		Mary		
		M1	M2	M3
George	A1	470	1,050	600
	A2	540	600	930
	A3	320	280	710

To determine the location of the restaurant, they decide that Mary will select a motorway (her options are M1, M2 and M3) and George a road (his choices are A1, A2 and A3), and that the intersection of both will be the site they will select. How should each make their selection in order to achieve the outcome that is in their best interests?

George is a pessimist and looks at the lowest of the three roads (470, 540, 280),

the minimums of each row, and decides to select the A2, which guarantees him a height of 540 metres. Similarly, Mary must evaluate the highest values for each motorway (540, 1,050 and 930) and decides to select the M1, which guarantees her a minimum height of 540 metres. Thus both make their selections and the result, 540 metres is the best for each. Put another way, if one of the two change their selection, the result will mean a worse outcome.

On the one hand, these examples show the variety of situations in which it is possible to find optimal solutions that are in the interests of two people (or groups) when they are completely opposed, and on the other, they show that if the payoff matrix has a saddle point, the result is strictly determined by the optimal choices of the two players.

When there is no equilibrium: mixed strategies

Many competitive games and the situations that are modelled by them cannot be resolved by means of pure strategies because they do not have a point of equilibrium. It is often the case that there is no dominant pure strategy for each player, that is to say, a strategy that remains the best one each time they make a move. Under these circumstances, the two players should not reveal their strategy and try to hide it, even trying to deceive their opponent. This is the case, for example, in poker, in which the players attempt to deceive their opponents and do not show their cards unless strictly necessary.

Determining an optimal mixed strategy

Let us recall the third and last game we considered in the first section of this chapter. Each player can write two numbers: player A can write either 1 or 8 and player B can write either 7 or 2. If the numbers written by both players have the same parity (both even or odd), A will win their value in pounds whereas if one is even and the other odd, B will win that amount.

The payoff matrix for the game is as follows:

	B writes 7	B writes 2
A writes 1	1	–2
A writes 8	–7	8

We can see that the game appears to have equal opportunities for both players (A can win £1 or £8, and B can come away with £2 or £7), and there is no saddle point: the maximin is -2, while the minimax is 1. Therefore, there will be no pure strategy for each player. Let's see if we can establish a mixed strategy that allows us to determine the value of the game. A mixed strategy requires a certain randomisation of a set of pure strategies. It is construced by assigning a probability to each pure strategy that relates to the frequency with which each pure strategy is used. For example, in our case A has two pure strategies (write 1 or write 8), and B also has two. The probabilities p (write 1), p (write 8) for A and p (write 7), p (write 2) for B,are used to maximise the player's potential. Knowing the odds and payouts assigned to each case, this will determine the expected value of the game.

First we must determine the probabilities A must assign his two pure strategies: Let's say the that p *is* the probability of writing 8 so $1 - p$ is the probability of writing 1. Thus if B opts for the strategy of writing 7, the expected value (V) for player A will be:

$V = 1\ (1 - p) + (-7)\ p$; this is a linear equation: $V = 1 - 8p$.

If on the other hand, B opts for the strategy of writing 2, the expected value for A becomes:

$V = (-2)\ (1 - p) + 8\ p$, which gives the equation: $V = 10p - 2$.

Player A wishes to determine p in order to obtain the highest possible expected value, regardless of the strategy chosen by B. Solving the system of equations gives the value of p and V for player A. In this case $p = 1/6$ and $V = -1/3$.

We can calculate the mixed strategy for player B in a similar manner. Let p be the probability of writing a 2, making the probability of writing a 7 $(1 - p)$. If A has opted for the strategy of writing 1, the expected value of B will be:

$V = 2p + (-1)\ (1 - p)$, which gives the equation: $V = 3p - 1$.

Similarly, if A has selected the other strategy, writing an 8, the expected value of B will now be:

$V = (-8)p + 7\ (1 - p)$, or rather $V = 7 - 15p$.

Player B wishes to determine p in order to obtain the highest possible expected value, regardless of the strategy chosen by A. Using this system gives the value of p and V for player B. In this case, solving the two equations gives: $p = 4/9$ and $V = 1/3$.

The method applied here can be generalised in a 2 x 2 matrix and solves games for which there is no saddle point by using mixed strategies. Let us now analyse the meaning of the results that have been obtained in greater detail. Firstly, it can be observed that the expected value is the same for both A and B ($V = 1/3$), with only the sign varying – for A the value is negative, meaning that A will lose, whereas for B it is positive, meaning that B will gain what is lost by A. In general, the value of the game (the average balance of A) is given by the expression: (ad - bc) / (a + d - b - c), where a, b, c, d are the values of the payoff matrix (left to right, top to bottom). Thus in our case, the value will be: (8 - 14) / 18 = -6/18 = -1/3, which shows that on average, A will lose £1 every three games, provided that both players play in line with their optimal strategy.

The mixed strategies determined both for A and B can also be found directly. Effectively, the proportion with which A must select one pure strategy or another is obtained by taking into account their gains or losses for each row. Specifically, making the calculations: 1 - (-2) = 3 (first row) and -7 - 8 = -15 (second row). Thus it becomes clear that their optimal strategy must be to play randomly with a ratio of 15 to 3, or rather 5 to 1 in favour of writing a 1. For example, throwing a dice with 5 faces marked with a 1 and one with an 8. Note that this result agrees with that obtained by solving the system of equations and finding that the probability of writing 8 must be 1/6 and, thus the probability of writing a 1 must be 5/6.

In a similar manner, player B, this time operating on the columns: (first column: 1 - (-7) = 8, second column: - 2 - 8 = -10), must play randomly with a proportion of 10 to 8, or rather 5 to 4 in favour of writing a 7 instead of a 2. This result is consistent with the system of equations solved above, which gives a probability of 4/9 of writing a 2 and thus, 5/9 of writing a 7.

It is now possible to formulate the optimal mixed strategy for each player: A will randomly choose between writing a 1 (with a probability of 5/6), or an 8 (probability 1/6). Likewise, B will randomly choose between writing a 7 (with a probability of 5/9) or a 2 (with a probability of 4/9).

THE MINIMAX THEOREM

For all finite, two-player zero-sum games, there is a value *V*, such that it represents the average value A expects to win from B if both play fairly, that is to say, playing to optimise their winnings.

Von Neumann, who devised and proved this theorem, considered the most relevant in game theory and applied in different ways throughout this Chapter, sensed that the result was feasible for three main reasons.

1. The existence of a strategy for the first player which is in their best interests and will allow them to obtain a determined gain (the average value of the game) and against which the second player can do nothing.
2. The existence of a strategy for the second player which is in their best interests, that is to say which will ensure that on average they do not lose more than a determined value (the average value of the game) and against which the first player can do nothing.
3. The fact that this is a zero-sum game in which the first player gains what the second loses implies that if there is an average value, both the first player and the second accept the respective gain or loss since any other strategy moves them away from this value and damages their interests.

Finally, even though there is still no saddle point, it is possible to ensure that if each player selects their optimal mixed strategy, on average B will win £0.33 per game. If B selects any other strategy and A does not switch, their gains will be reduced. However if they maintain their optimal mixed strategy and A changes their strategy, A's losses will increase.

Applications of mixed strategies

The previous section has provided an in-depth example of how a game is solved by determining optimal mixed strategies for each player in cases where the analysis of the payoff matrix shows there is no saddle point for the game – that is when the minimax and maximin do not coincide. To avoid distracting the reader, the example used an abstract game that made it possible to focus on the values of the payoff matrix, without taking into account other issues regarding their meaning.

Let us now consider another example to see how the method could be applied in the real world.

Growth of a company

A company has developed a new product and is evaluating its launch on to the market for the coming year. It can decide between reduced production, assuming that the performance of the economy will be poor, or large-scale production, assuming the economy will recover in expectation of strong sales. The expected profits (in thousands of pounds) are given in the following table:

		Performance of the economy	
		Bad	Good
Quantity produced	Small	500	300
	Large	100	900

In order to make their decision, the company management assume that the behaviour of the economy will be in line with a mixed optimal strategy. What is their optimal mixed strategy and the expected payoff?

The values of the matrix show that there is no pure optimal strategy, since there is no saddle point (maximin = 300, minimax = 500); thus it will be necessary to determine an optimal mixed strategy.

Let p be the probability of large-scale production, $(1 - p)$ be the probability of small-scale production, and V be the expected value. Thus if the economy performs badly, the expected value will be:

$$V = 500\ (1 - p) + 100\ p, \text{ which can be rewritten as: } V = 500 - 400\ p.$$

In contrast, if the economy performs well, we will have:

$$V = 300\ (1 - p) = 900\ p, \text{ or rather, } V = 300 = 600\ p.$$

Solving the system gives: $p = 1/5$ and $V = 420$. This result means that if the situation can be repeated a large number of times, the optimal mixed strategy will be to use the 'large-scale' production strategy 1/5 of the time on a random basis, and the 'reduced' production strategy 4/5 of the time, with an average expected profit of £420,000.

$V = (ad - bc) / (a + d - b - c)$, where a, b, c, d are the values of the payoff matrix (left to right, top to bottom). In this case we have: $500 \cdot 900 - 300 \cdot 100) / (500 + 900 - 300 - 100) = 420{,}000 / 1{,}000 = 420$, clearly consistent with the result based on the system of linear equations solved above.

In addition, the problem has been solved under the assumption that the behaviour of the economy will also follow an optimal mixed strategy. The calculation for the economy indicates that the probability of good performance is 2/5 and thus poor performance will arise with the probability 1 - 2/5 = 3/5.

Taking a penalty

In a football match, taking a penalty kick can be interpreted as a competitive game between the striker and the goalkeeper, in which both contenders have opposite interests. Let us suppose that the striker can kick the ball right, left or into the centre (these are the three pure strategies) and the goalkeeper can dive to their left or right, or remain in the centre (these are also pure strategies). Combining the statistics of performing well and making a mistake for both the striker and the goalkeeper, the following table has been drawn up:

		Goalkeeper (B)		
		R	C	L
Striker (A)	R	0.9	0.9	0.6
	C	0.8	0.1	0.7
	L	0.5	0.8	0.8

Each square in the table gives the probability of a goal being scored (the striker wins) in line with the strategies adopted by both players. For example, if the striker directs their kick to the right and the goalkeeper dives to the right (both go in opposite directions) the probability of a goal being scored is 0.9, whereas if the striker directs their kick to the centre and the goalkeeper remains in the centre, the probability decreases to just 0.1. Under these circumstances, what strategies

THE RAND CORPORATION

The RAND (Research and Development) Corporation is a US-based 'think tank' formed at the end of the Second World War, initially in order to conduct research of a primarily strategic nature for the United States Air Force. In spite of the secret nature of its projects and the questionable goals of many of them, it cannot be disputed that at its heart lie some of the best scientists working on game theory. Thanks to this company – by 1948 it had acquired the status of a private company working exclusively for the air force – it was possible to undertake the basic research that proved fundamental to the development of the theory.

Its internal organisation had more in common with a university research institute that a military structure. During the 1950s and 1960s, together with applied research, some related to nuclear arms and the start of the cold war, fundamental research was conducted by the most distinguished mathematicians and economists in the field of game theory, including von Neumann, John Nash, Merril Flood, Kenneth Arrow and many more, all of whom were active at RAND around the same time, a short period during which the first great developments of game theory were made.

The new site of the RAND Corporation in Santa Monica, California.

should be adopted by the striker and the goalkeeper?

An initial analysis of the problem shows that there is not a dominant pure strategy and that it is not possible to solve the situation using pure strategies, since the maximin is 0.6 and the minimax is 0.8, or rather the striker is expected to score six out of ten kicks whereas the goalkeeper is expected to let a goal in 8 times for every 10 penalties. Both want to (and can) improve their results (gains): The striker can increase their probability above 0.6 and the goalkeeper reducing the probability from 0.8.

The optimal mixed strategy of the striker and the goalkeeper is calculated, as well as the average value of the game, which in this case has a value between 0.6 and 0.8, indicating the average number of times the striker will score a goal when taking the penalty kick.

The optimal mixed strategy of the striker is obtained by calculating the probabilities of selecting each of the pure strategies, that can be referred to as: $p(r)$, $p(c)$, $p(l)$. Given that $p(r) + p(c) + p(l) = 1$, this can be reduced to two probabilities: $p(r)$, $p(c)$, $1 - p(r) = p(c)$. As always, let V be the expected value of the game.

If the goalkeeper dives to their right, the expected value for the striker will be

$$V = 0.9\, p(r) + 0.8\, p(c) + 0.5\, (1 - p(r) - p(c)).$$

If the goalkeeper remains in the centre, this will be:

$$V = 0.9\, p(r) + 0.1\, p(c) + 0.8\, (1 - p(r) - \mathrm{p}(c)).$$

And if the goalkeeper dives to their left, we have:

$$V = 0.6\, p(r) + 0.7\, p(c) + 0.8\, (1 - p(r) - p(c)).$$

This gives us a system of three linear equations with the following solution: $p(r) = 0.37$, $p(c) = 0.19$, $p(l) = 1 - p(r) - p(c) = 0.44$, and the value of the game for each player is $V = 0.71$.

It is also possible to calculate the probabilities of the goalkeeper selecting each of the three pure strategies in a similar manner, however this has been left up to you.

Advantages and drawbacks of the minimax method

It is clear that the minimax theorem and the general method that has been set out so far, both in the case of pure strategies and in the case of mixed strategies that rely on chance, are powerful tools for solving matrix games in order to obtain the best possible results. The theorem has many applications in diverse areas such as economics, politics, sports and military conflicts, as we have begun to see. It has been possible to solve not only scenarios that have dominant strategies or have a point of equilibrium, but also in examples that lack a point of equilibrium where it is possible to find the average value of a game, optimising the gains of both players through the use of mixed strategies.

However, in all cases we have assumed a condition referred to as the 'fair play' of both players. This principle assumes that each player will suppose their opponent will always act in favour of their interests and will apply the fairest strategy to ensure this is the case. However, what happens if this is not the case and one of the two players attempts to trick their opponent?

Morton Davis' introduction to game theory explains how various researchers carried out a number of experiments between the 1950s and the 1970s in order to observe the behaviour of different players when playing matrix games. Specifically, in 1964, Richard Brayer devised a game that could be solved using pure strategies, that is to say, with a point of equilibrium that could be easily calculated. He told the players that they would sometimes be playing against an expert and at other times against a player who chose his strategies at random, when in reality they were always playing against a researcher who was frequently changing strategy. On some occasions they would use the optimal strategy B, however on others they would play at random. The game has the following payoff matrix.

		Researcher		
		A	B	C
Player	a	11	–7	8
	b	1	1	2
	c	–10	–7	21

The game can be quickly solved by applying the minimax theorem, since the point of equilibrium is 1, square (b, B) of the matrix, meaning that the player must choose move b and the researcher B, with a gain of 1 for each player in each game.

The experiment showed that the players played with strategy b when they saw the researcher repeatedly used strategy B. However by contrast, when their opponent played at random, they changed, tending to select strategy a in order to maximise their gains while accepting the risk of incurring losses. Post-interviews showed that more than half of the players thought that it was 'stupid' for the researcher to systematically choose strategy B, since this was to accept a loss of 1, when, by choosing other strategies, they could 'perhaps' improve their results, without acknowledging that if the player kept playing with strategy b, the loss of at least 1 by the researcher was guaranteed.

Both this study and other similar ones on the behaviour of players showed that playing fairly to optimise gains is not a normal thing to do, and that people tend to prefer strategies that apparently provide them with greater gains and will only switch to the optimal strategy when they repeatedly verify that this is not the case. Even more chaotic is the behaviour of players when the game does not have a point of equilibrium – where it is necessary to apply mixed strategies. In this case, even knowing the method, the majority do not deem it necessary to make calculations and play according to their intuitions, which generally differ from the optimal mixed strategy.

All these experiences show that when we are confronted by reality, we must doubt assumptions of 'fairness', such as assuming that our opponent will play in the smartest manner and in line with their interests. Perhaps the explanation of this phenomenon is to be found in the fact that the minimax strategy is a defensive one: it guarantees certain results, the best possible ones if one's opponent plays in the smartest way. However, if we leave this assumption to one side, why should a player not attempt to improve on the results?

This Chapter has analysed two-player competitive zero-sum games and concluded that for this type of game there exists an optimal strategy for each player and a game value that makes it possible to determine the average gain for each player. The information for this type of game can always be represented using a payoff matrix in which each row indicates a strategy for the first player and each column indicates a strategy for the second. The process that can be followed for solving a two-player zero-sum game can be summarised as follows: calculate the maximin

(maximum of the minimums) for the first player and the minimax (minimum of the maximums) for the second. If both are equal, this means that the optimal strategies for each player give the same value (the value of the game) and it has been solved. In this case, the strategies for each player are called pure strategies.

If the maximin and minimax are different, the selected strategies (pure strategies) are discarded and all strategies for each player are reconsidered, assigning a probability to each one. The value of these probabilities (the sum of which must be 1) will determine an optimal mixed strategy and give the average value of the game for each player.

The determination of the probabilities and the average value for each player should be calculated by solving a system of linear equations (the number of equations depends on the number of strategies), the unknowns of which are the probabilities being sought and the average value of the game. If the average value is the same for both players, the game is solved and the probabilities that have been obtained for each player determine their optimal strategy, which will be a mixed strategy, due to its random nature.

In the event that the average values of the game do not agree for each player, or rather, upon calculating one of the probabilities it is negative, the game is not solved. In this case it will be necessary to go back and analyse the game to see if there is a dominant strategy; otherwise, the method cannot be applied in that case.

Chapter 5

The Game of Life: Theoretical Applications in the Real World

Competition is the mother of science… and life. Competition and cooperation make us who we are..
Erwin Neher, Nobel Prize for Medicine winner

All the situations presented in the previous chapter refer to purely competitive games. The winnings of one player were always equivalent to the losses of the other, hence the term 'zero-sum game'. These are situations of total conflict, where the players' objectives are completely opposed and each attempts to maximise their gains and in so doing maximise the losses of their opponent.

However things are different in this chapter. Although the player's objective is still to win and the game is a conflict situation, it is no longer a total war. Firstly, the winnings of one player no longer correspond directly to the losses of the other, and there are even certain strategies in which both can be considered as winners. On the other, there are situations in which cooperation can bring benefits to both parties. This requires the introduction of elements of communication and mutual trust, but also implies threats that ensure compliance with any agreements that are made. Under these circumstances, we can talk of situations of *partial conflict* and distinguish between *cooperative strategies* and *non-cooperative strategies* (note the term *to desert* is often used to refer to a strategy that is the polar opposite of cooperation).

Remember that game theory is focused on decision making and this aspect now becomes more important than ever before. In many of the situations we shall consider in this chapter there is a tension between competing and working together. What are the decisions available to each player under these circumstances? This suggests what can be understood as a 'dilemma'. Both players can work together or compete and it is unclear which decision will provide the greatest benefits since everything depends upon the decision of the opponent. In general, mutual cooperation will result in benefits for both, and is the best overall outcome, whereas mutual confrontation will lead to disaster. If there were only these two

possibilities, there would be no dilemma. However the problem arises when one of the players attempts to cooperate and the other opts for confrontation. This results in increased benefits for this perfidious player, greater than could have been obtained by cooperating. The dilemma is clear.

The complexity of games of these types means that this chapter must necessarily mix mathematics with psychology and even morality, meaning that the solutions are often not strictly mathematical. Instead they are presented as two possibilities that are dependent on the decisions of the players. However these games are more interesting than the pure conflicts discussed in the previous chapter, since they come up much more frequently in the real world, where it is uncommon for dispute to not have a mixture of confrontation and cooperation.

It is possible to view the array of two-player situations that game theory attempts to analyse as a spectrum running between two extremes. At one end are the purely competitive zero-sum games, while fully cooperative ones form the other extreme. Both types are, at least in theory, easy to solve. We have seen this in the competitive situations from the previous chapter and the same can be said of situations of pure cooperation. An example of when players have the same objective would include the driver of a rally car and his navigator, a pair of dancers, and an aeroplane pilot and an air-traffic controller. The way of solving these games involves joining forces (coordinating moves efficiently) in order to achieve the objective.

The remainder of two-player games covered in this chapter are somewhere in between the two extremes. These are more complicated because the participants have both conflicting and shared interests even though this may not always seem to be the case. As an example, consider a person selling an apartment and the potential buyer. Both are interested in reaching an agreement (cooperation), although their interests differ when it comes to price (conflict). Other examples include the merger of two companies, or even two countries at war, a situation in which the majority of strategies are conflicting, but in which it may be possible to agree on cooperation or a pact, however partial, such as agreeing a truce or not to use nuclear arms.

The mathematics of cooperation: non-zero-sum games

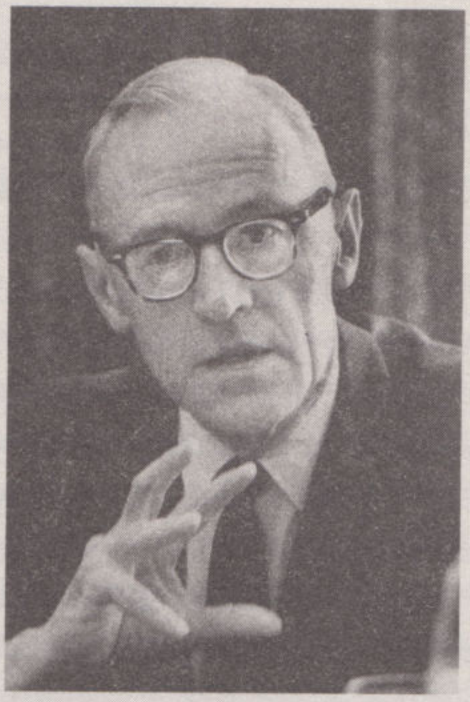
To illustrate the difference between zero-sum games and non-zero-sum games, let us consider a situation related to the screening of an advert. Two companies of the same type, A and B, wish to promote their products and both receive an offer from

THE DEVELOPMENT OF GAME THEORY

In 1955 von Neuman and Morgenstern published a work that laid the foundations of game theory, by describing a method for finding optimal solutions for two-player zero-sum games. From then on research in the field began to focus on cooperative games and the analysis of optimal strategies for situations in which participants could establish agreements about the most appropriate strategies.

A significant development in game theory took place in the 1950s with the appearance of the first theorisations of the prisoner's dilemma. The decade also saw John Nash define the concept of an optimal strategy for games with multiple players where the optimal strategy cannot be established beforehand. This is still known as Nash's equilibrium and can be applied to non-cooperative games, although it may also be extended to cooperative ones.

It was also at this point that the first applications of game theory in other fields beyond economics, such as philosophy, science and politics, began to surface. Later on, in the 1970s its applications would be extended to biology, largely thanks to the work of John Maynard Smith, who introduced the idea of the evolutionary stable strategy.


Oskar Morgenstern was one of the inventors of game theory along with John von Neumann.

a television channel. They can advertise in the afternoon (40% of this channel's viewers are watching in this time slot), or at night (with 60% of viewers), but not in both slots, and it is known that there is no overlap between the two. If the two companies screen their adverts in the same slot, each will make sales to 30% of the viewers of that slot and none to the other, whereas if they screen their adverts in different slots, each will capture 50% of the audience of their slot. Which is best for each company? Does it help to discuss the decision with the other company or is it best to keep it secret?

The game can be expressed using a payoff matrix in which each value indicates the percentage of sales captured by each company. However, it is no longer possible to have a single value for each entry in the matrix, since what is won by one company

is not what is lost by the other, but the profits of each company. Thus two values are used, the first of which will be the profits for A and the second, the profits for B, in line with the strategies they adopt.

		Company B	
		Afternoon slot	Night slot
Company A	Afternoon slot	(12,12)	(20,30)
	Night slot	(30,20)	(18,18)

If A and B both screen their advert in the afternoon, each company will capture 12% of the total audience (30% of 40%), however if they show their advert at different times, the results will be symmetrical. Thus if A opts for the afternoon slot and B the night one, A will obtain 20% (half of 40%) and B 30% (half of 60%) of the total audience, and if both switch strategies they will also switch their corresponding gain.

In order to analyse the game like we did in previous examples, we should now consider two matrices (the profits for each player), assuming that each attempts to maximise their profits in line with the payoff matrix.

MATRIX FOR PLAYER A

		Company B	
		Afternoon slot	Night slot
Company A	Afternoon slot	12	20
	Night slot	30	18

MATRIX FOR PLAYER B

		Company B	
		Afternoon slot	Night slot
Company A	Afternoon slot	12	30
	Night slot	20	18

Given the symmetry of the two matrices, and taking into account that the strategies for A correspond to the rows and the strategies for B the columns, the analysis is similar for both. It can be carried out in the same way as for a zero-sum game. There is no saddle point (maximin 18, minimax 20), meaning that it is necessary to find a mixed strategy that gives the value of the game for player A. This strategy consists of using strategy 1 (afternoon advert) with a probability of 3/5 and strategy 2 (night advert) with a probability of 2/5, giving a value of 19.2 (average profit per game). Similarly, on account of the symmetry, B will follow a matching pattern. They will randomly play twice using strategy 1 and three times using strategy 2 every 5 moves, thus obtaining the same average profit. Up to this point, everything would appear to be as it was before, and we could be forgiven for believing this to be the optimal strategy for each player – the game is solved.

However, a more detailed analysis of the game shows that in this case, each of the two players can aspire to increase their profit without affecting those of the other. As such, the previous solution is not an optimal one, and the value of the game obtained using the mixed optimal strategies for zero-sum games is not always the highest possible.

This occurs because the optimum strategy of zero-sum games is based on the idea of limiting (or reducing) the maximum winnings of one's opponent, which, for zero-sum games, means increasing one's own profits as much as possible. However, this is no longer the case. Suppose that instead of using a mixed strategy, company A decides to opt for pure strategy 2 (night slot), while company B uses the mixed optimum strategy. Under these circumstances, A will make, on average: $30 \cdot 2/5 + 18 \cdot 3/5 = 22.8$, whereas B will continue to make 19.2. Note that while B continues to make the same profit, A's profit has increased, a situation that is impossible in zero-sum games. It is clear that B may wish to do likewise,

using pure strategy 2 and expecting A to use a mixed strategy. It is now B who increases their profits, without reducing those of A.

However, what happens if both companies use pure strategy 2? Then both obtain only 18% and reduce their profits by the same value. This would appear to be a dead end then, since each company can increase their profits without affecting the other, although if both try to do so, not only do they fail, but they make less than the average expected value.

Yet there is another possibility. Let us assume that both players agree not to select the strategies with which they make the least profit (i.e. both screening their advert in the same slot). Under such an agreement, the two companies can earn much more, and can even do so in a way that their profits will be the same. Af A alternately selects between strategies 1 and 2, and B alternately selects between strategies 2 and 1, the average profit for both companies will be 25 per game (A's profits alternate between 20 and 30, whereas B's alternate between 30 and 20). Thus, this would appear to be the best solution, and it is also an equilibrium solution.

A fair idea: Nash's equilibrium

After their study of two-player zero-sum games, von Neumann and Morgenstern expanded their research to games with more than two players, taking into account possible partnerships (groups of two or more players agreeing to work in a coordinated manner), thus moving away from strictly competitive games. In the 1950s, John Nash extended the theory to non-cooperative *n*-person games, in which partnerships were forbidden. Nash was principally concerned with non-zero-sum competitive games for two or more players and came to establish the concept referred to as Nash's equilibrium.

His method is apparently simple, at least in terms of its main idea. Let us assume that various players have just made a move and each has selected a certain strategy. Once the result of the game is known, each player is asked if they believe the way in which they have played to be satisfactory, or to put another way, if they would have preferred to act differently. If the reply is affirmative, that is to say, if all participants believe that they have chosen their best strategy, the result of the game is a point of equilibrium similar to that described by Nash.

Let us consider the application of this idea to a specific case. The following matrix provides the results of a non-zero-sum game.

	Strategy 1	Strategy 2
Strategy 1	(1,100)	(0,1)
Strategy 2	(2,0)	(5,2)

Two players select strategy 2. Once the result is known, both agree on their strategy and believe it is the best they could have done. The first player (strategies in rows) believes that they have won 5, which is the maximum they can obtain, while the second, when they know that the first has selected strategy 2, also agrees with their selection, since they have won 2 instead of winning nothing.

However the previous solution can be disputed. One could argue that even if the first player's selection is 'good' because the strategy opted for (2) is dominant, at some point it will occur to the second player that selecting strategy 1 could have allowed him or her to win 100. However, in a competitive game in which each player is focused on maximising their gains, this result will never occur if it is assumed that player 1 chooses strategies rationally.

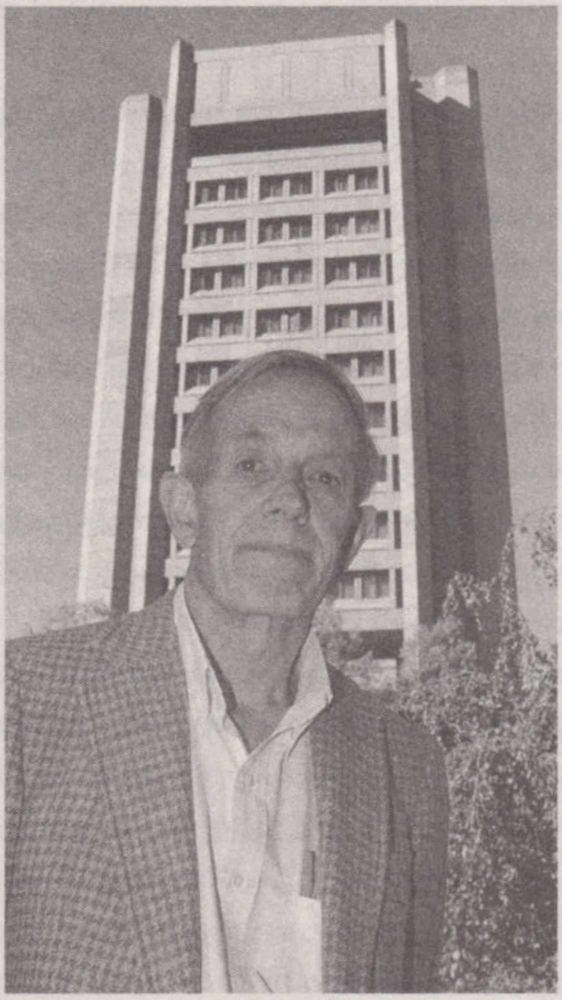
As such, from the four possible results, the only one in which neither of the two players regrets their move is (5,2). This result is a Nash equilibrium. In any game with a different result, one of the players would object to their own way of playing because, in the words of Nash, it would give an unstable solution.

The method applied to obtain the previous solution appears interesting and provides a rational solution. In this context, Nash showed that any finite two-player game has at least one point of equilibrium, thus extending von Neumann's minimax theorem. In zero-sum games, the point of equilibrium coincides with that obtained using the minimax theorem, however Nash's result is interesting because there are also points of equilibrium in non-zero-sum games, as we have seen in the previous example, and the solutions are still fair.

However the same thing does not always happen. Sometimes the solution provided by the point of equilibrium is surprising and has strange properties, in spite of appearing to be wholly rational.

JOHN FORBES NASH (1928)

After the work of von Neumann, the contributions made by John Nash – especially his early work – are possibly the most significant in game theory's short but intense history. As a child, he showed great intellectual ability, although there were also signs that he struggled when it came to interacting with others. Although he began by studying chemical engineering, he soon switched to mathematics, a subject in which he was especially gifted. In 1948 he received a scholarship from Princeton University, where Einstein and von Neumann were working at the time, to study for a PhD in game theory under the supervision of Albert W. Tucker. He submitted his thesis in 1950, a short but highly original work on non-cooperative games that quickly became well known among game theory experts. He created a connector game, currently sold under the name of Hex and based on a board of hexagonal tiles. Although he was unaware of the fact, his game was similar to one that had been created by the Dane Piet Hein a few years earlier. However Nash showed that there must be a winning strategy for the first player, even if the strategy was not known.



From the 1950s, he worked at the Massachusetts Institute of Technology (MIT) and the RAND Corporation, a famous organisation that was part of the US air force and concerned with strategic issues. Shortly after his marriage in 1959, he was taken into care as a result of schizophrenia, which he had been developing for quite a while and which has persisted throughout his life. However, in spite of this, he continued to work on game theory until 1994, when he received the Nobel Prize for Economics.

In 2001, the director Ron Howard created the film *A Beautiful Mind*, which won four Oscars and whose story was based on the biography of John Nash, focusing specifically on his battle with the mental illness he suffered throughout his life.

Prisoner's dilemma and other classic problems

The examples from the previous section have shown it is sometimes possible for non-zero-sum games to use cooperative strategies to improve the results. The problem arises when this improvement is not equally distributed among the participants. Put another way, the problem is one of how to distribute 'the excesses' and whether all participants can be convinced of a rational way of doing so.

Merrill Flood, who worked at the RAND Corporation, analysed a variety of situations from everyday life, especially those in which players were required to distribute any additional gains. One such situation was the sale of a used car. One person wants to purchase a used car that his friend is willing to sell him. To agree a price, the two go to a second-hand car dealer to have the car valued. The dealer is willing to purchase the car for £1,000 and will resell it for £1,300, earning a minimum of £300 for the transaction. Conducting the sale directly, without the involvement of the dealer, it is clear that they will save £300. The pair could decide to split the money equally, such that the sale will be made for £1,150, with each pocketing £150.

While this may appear to be the most rational solution, it is not the only one. One of the participants in the game, for example the buyer, may decide that they are unwilling to pay more than £1,100, meaning that the seller, if they accept, will still make £100 over the price at the dealership. On the other hand, the seller may decide to set a minimum price of £1,250 with the argument that the buyer is still saving £50. Note that if one of the two rejects the other's offer with the rational argument that the distribution of the extra money is 'unfair', they are losing out too, since the price is still lower than what they would have to pay to the dealer.

However, the idea of a fair distribution of gains is not always so clear, and on some occasions there may be more than one solution that can be considered as completely fair. Suppose that Geraint needs to drive from Cardiff to London (200 km) for a work meeting, and will return the following day. He finds out that Trevor, one of his friends who lives in Swindon, also needs to go to London on the same day and they decide to share the car, both on the way there and on the way back. Given that Swindon is half-way between Cardiff and London, how should the pair split the costs of the journey?

Argument 1: Since the route travelled by Geraint is double that travelled by Trevor, the cost should be divided by three, with Trevor to pay one third and Geraint the remaining two thirds.

Argument 2: given that Geraint will travel on his own for half the journey whereas both will travel together for the other half, Geraint should pay the full cost of his half plus half of the other part, meaning that Trevor will only need to pay half of half, or rather a quarter. Thus, dividing the cost by 4, Trevor will pay one part and Geraint the remaining three.

When it comes to dividing the money, let us note that it will cost Geraint £60 to travel from Cardiff to London if he travels on his own and it will cost Trevor £30 to get from Swindon to London. If both travel together, they will save £30 between them. Based on the first argument, Geraint will pay £40 (saving £20) and Trevor will pay £20 (saving £10). On the other hand, based on the second argument, Geraint will pay £45 (saving £15) and Trevor will pay £15 (also saving £15). Thus the second argument corresponds to sharing the savings equally, whereas the first corresponds to sharing them in proportion to the expenses envisaged for each. Even when thinking rationally, there may be more than one fair solution.

The prisoner's dilemma

The prisoner's dilemma, one of the best known problems from game theory, is a type of non-zero-sum game devised by Albert W. Tucker in 1950. It is a simple example of a phenomenon that occurs in many situations where there is a conflict between two forces that can opt to confront each other or cooperate, such as in price wars, advertising campaigns or arms races.

Although the name of the dilemma makes reference to a prisoner and can be formulated as a game played by two criminals unsure of whether to plead innocent or guilty – and in so doing implicate their opponent – we shall see it at work in one of its most interesting applications, namely military conflict, where the losses and gains arising from the 'game' equate directly to human lives:

> Two powers, P1 are P2 are in dispute and must decide on their arms policy.
> Each can choose between two independent strategies:
> A: refuse to cooperate and arm itself as if preparing for a possible war.
> B: cooperate and disarm or at least agree to the prohibition of certain arms.

The four possible results, (A,A), (A,B), (B,A) and (B,B), in which the first coordinate is the strategy of P1 and the second that of P2, can be expressed in a table

ALBERT WILLIAM TUCKER (1905–1995)

Tucker made important contributions to topology, non-linear programming and game theory. He graduated in mathematics from the University of Toronto and completed his PhD research at Princeton University in 1932. After various spells at the universities of Harvard, Cambridge and Chicago, he returned to Princeton, where he taught as head of the mathematics department for over 20 years until 1970. In 1950, he named and gave the first exposition of game theory's most famous and interesting paradox, the prisoner's dilemma, which made a fundamental contribution to the model of conflict and cooperation developed by M. Flood and D. Dresher at Princeton.

In addition to his significant research work, he was also a gifted teacher who took an interest in mathematical education. He became involved with projects for secondary school education which resulted in him becoming president of the Mathematics Association of America (MAA). His many PhD students included the Nobel laureate, John Nash.

		Power P2	
		Option A	Option B
Power P1	Option A	(A,A) Arms race	(A,B) Only P1 arms
	Option B	(B,A) Only P2 arms	(B,B) Control of arms, or disarmament

It is also possible to assign values (numerical payoffs) to the results of mixing the various strategies, bearing in mind that in this case, the payoffs will vary for each player and as such there will be two numbers in each square, the first corresponding to the winnings of P1 and the second to the winnings of P2.

Thus we will have the following payoff matrix:

		Power P2	
		Option A	Option B
Power P1	Option A	(2,2)	(5,0)
	Option B	(0,5)	(4,4)

If the figures are interpreted as winnings, the dilemma is clear. What should P1 do? Regardless of the option chosen by P2, it is in P1's best interests to arm itself. If P2 chooses A, P1 will win 2 if it decides to arm itself and 0 if it does not, whereas if P2 selects B, P1 will win 5 if it arms itself and 4 if it does not. The results are symmetrical for P2, which also means that it is better to arm itself , regardless of the two possible strategies for P1. Thus it can be said that the solution (A,A), whereby both powers arm themselves, with a payoff of 2 for each, is the non-cooperative equilibrium solution towards, which it appears the game is directed.

However, it is better for each power if the other disarms (the gains are higher) and in addition to this, the maximum overall gain is obtained when both powers disarm. As such, if both powers do not cooperate, the best overall result (4,4) is impossible. However if one power chooses to cooperate, they assume a grave risk given that they do not know the action being taken by the other. They will obtain the least gain if the other power does not cooperate and hence trust becomes a vital element in the game. Without it, the best result is completely unstable, because each power will attempt to protect itself against possible non-cooperation by their opponent.

There are many other real-life situations, which in general are less extreme than the one set out above, and in which it is possible to find situations where cooperation, although difficult, is feasible. Games are often repeated a number of times and thus important factors, such as reputation and trust can become highly significant, meaning that players can come to realise the existence of mutual benefits. In our example, disarming evidently has many advantages when faced with an unchecked arms race which, in addition to the high costs involved, could eventually lead to total disaster. However, cooperation only works if carried out long term.

Although the formulation of the prisoner's dilemma corresponds to game theory, the problem that lies at its heart is much older. Thomas Hobbes (1588–1679), the

English political philosopher and author of *Leviathan*, analysed a situation similar to this dilemma in terms of social evolution in his theorisation of political absolutism. Hobbes claimed that society's natural state was one of anarchy in which only competition was important. In order to make cooperation possible, he believed it was necessary to impose restrictions and ensure they were complied with. Hobbes saw the social contract as the imposition of a cooperative result and believed that society must submit itself to the arbitration of the government, since the decisions that implied competition or cooperation could not be left in the hands of individuals.

In the business world it is also possible to find various situations in which scenarios similar to the prisoner's dilemma arise. In a competitive industry it is often the case that participants abstain from certain practices that would allow them to gain an advantage over others with the logic that in the long term. This will be beneficial for all, particularly on an individual level. This is the case when it comes to agreements made by book stores to restrict discounts to a certain level (e.g. 10%). These are sacrifices

ROBERT AXELROD AND THE PRISONER'S DILEMMA.

Robert Axelrod, professor of public policy at the University of Michigan, mathematician and doctor of political science, is an expert in problems of cooperation, specialising in games such as the prisoner's dilemma. His works include *The Evolution of Cooperation*, an evolutionary study of cooperation with the main argument that the strategies people use tend to evolve towards more effective ones in which cooperation is required.

With respect to the prisoner's dilemma, Axelrod remarks that if the game is only played once, it is not possible to discover the behaviour of the other player, return their cooperation or punish their desertion, meaning that only short-term objectives are required. On the other hand, when the game is played repeatedly, it is possible to ground strategies in previous interactions, basing them on reciprocity. If the opponent has cooperated a lot, perhaps it is worth regularly cooperating. However if they have failed to do so, it is not even worth trying.

Given that the optimal strategy appeared to be unknown, Axelrod organised a tournament in which a number of eminent game theorists took part to allow him to observe how they played and attempt to find effective strategies. At the end of the game, out of all the strategies that had been tried, the best, referred to as 'eye for eye', was the most straightforward. The strategy was based on initially cooperating (never being the first to desert), and then playing based on the action taken by the opponent in the previous move. If the opponent cooperates, it is worth continuing with cooperation; if they do not, disagreement should be shown quickly.

that are carefully considered by all parties in order to improve sales, because all know that when one does not apply the measure, the others will refrain from doing so too, losing out on any additional benefit and in fact increasing their costs.

The game of chicken

Together with the prisoner's dilemma the so-called game of chicken is one of the most representative from game theory when it comes to non-zero-sum games. Its name is a reference to the metaphor for cowardice, and the game is often used to describe a challenge between two people faced with a risky situation in which one will be the first to give way before their rival.

A common formulation is as follows: two drivers are driving towards each other at high speed and each must decide at the last moment if they will turn to the right to avoid the collision. The following cases can arise:

1. Neither player turns and there is a collision. This is the worst result and the value 0 is assigned to both players.
2. The players turn at the last minute, avoiding the collision. This is a good result and is the same for both, although both lose 'pride' and neither is deemed to be the winner. The value 3 is assigned to each player.
3. One of the players turns and the other does not. The first will lose a lot of 'pride' and will be assigned a value of 1, whereas the other is deemed to have won the game and will be assigned a value of 5.

The various strategies and corresponding payoffs can be summarised in the following matrix:

		Driver 2	
		Turns	Does not turn
Driver 1	Turns	(3,3)	(1,5)
	Does not turn	(5,1)	(0,0)

Analysing the situation shows that if both contestants attempt to achieve their maximum payoff, refusing to turn in order to achieve the value 5, both will end up with the worst result. It would seem better to turn, since by doing so both will obtain

THE GAME OF CHICKEN

Although situations as extreme as the one suggested in this game do not frequently occur in real life, there are conflicts in which both players wish to dominate the situation (workplace relationships, conflicts between powers), which reach limit situations similar to the ones suggested by this game.

They arise even more often in fiction, such as in Nicholas Ray's film *Rebel Without a Cause* (1955), in which two players drive their cars towards a cliff, with the first to jump out losing the game of chicken.

Both the prisoner's dilemma and the game of chicken are games of partial conflict, which show that, on certain occasions, each player pursuing their immediate interests would lead to catastrophic results overall, and hence in this respect the games are similar. However, there is one aspect in which they differ. While in the prisoner's dilemma coincidental strategies provide the best results, in the game of chicken it is the other way round and doing the opposite of one's opponent always gives a better result than following the same strategy. This means it is better to show disagreement quickly.

a favourable result, although neither of the two wishes to turn before the other as this will mean they will obtain a payoff of 1, compared to 5 for their rival.

The game can be analysed from the point of view of cooperation, where turning is understood as cooperating and failing to do so as deserting, in such a way that if both players cooperate, the overall result is good. However, perhaps the most significant aspect is that the game represents a form of negotiation in which each of the participants attempts to put off making the concession required to avert disaster until the last moment in order to force the other into playing 'fairly' (in this case by turning).

Another aspect that defines this game is the role played by the convincing declaration of the strategy to be used before beginning the game, such as deciding to lock the steering wheel of one car in order to avoid it being able to turn, as a measure that forces the other into choosing an alternative strategy, that is to say ensuring that they turn in order to avoid an otherwise certain collision.

Both this game and the prisoner's dilemma show the difficulty of finding a solution to this type of situation in which both conflict and cooperation are possible. What is even more unsettling is the fact that the games reveal the conflict that often arises between our own immediate interests and those of the group.

Cooperate or die: The hawk-dove game

The different games analysed by game theory can be applied to a wide range of situations. They are generally to be found in economic, political and military situations, as their development was initially grounded in these areas. However, over time, they have also come to be applied in other areas, such as theories of evolution and ecology.

It is often assumed that decision making is the exclusive preserve of rational beings and, as such, game theory can only be applied to human behaviour. However, in 1978, excellent research by John Maynard Smith showed that it could also be applied to the behaviour of certain species, which adopted collective strategies for survival or to improve their development. The fight for survival can be understood as a process of competition in which certain behaviours by individuals would risk the disappearance of others. Moreover, the 'altruistic' behaviour of certain members of a group can be beneficial to the collective but fatal to the individuals in question.

John Maynard Smith suggested what is now known as the 'hawk-dove' dilemma, which in some senses is an application of the game of chicken. When two animals compete for prey, it is normal for both to display aggressive attitudes and use force to defeat their adversary. If the confrontation remains unresolved and is about to turn into a fight, there are two opportunities: abandon and flee (doves), leaving the prey but staying alive, or fight (hawks) with an unpredictable outcome that may result in death.

Let us suppose that a small group of 'hawkish' individuals lives among 'dovish' ones. The hawks will initially prosper as their strategy is beneficial (each time they come into conflict with a dove, they will win), meaning that over time, the number of hawks will increase. However this also means that conflicts between hawks will increase and that will result in deaths and a reductions in their numbers. Over the course of time, this situation will lead to an equilibrium between hawks and doves which, as we can see, is the case in the real world.

Smith made use of these circumstances to construct a game, assigning payoffs to the various actions, which can be represented in the following matrix:

	Hawks	Doves
Hawks	(–5,–5)	(10,0)
Doves	(0,10)	(2,2)

The payoffs assigned are based on the following schemes: achieving the objective (such as killing prey or securing a mate) 10 points; suffering wounds -20 points. In a conflict between hawks, assuming that while one hawk wins another loses, the average score is -5. When a hawk confronts a dove, they always win (10) whereas

JOHN MAYNARD SMITH (1920–2004)

John Maynard Smith was an English evolutionary biologist and geneticist who used mathematics and particularly game theory for his research on evolution. He attended school at the famous Eton College and studied engineering at Trinity College, Cambridge. From an early age, he was a member of the Communist Party, which he abandoned in 1956 after the invasion of Hungary by the Russian army. He soon changed his scientific direction and studied genetics at University College, London. He taught zoology at the same institution, and in 1958 published a popular science book, *The Theory of Evolution*, which grew to enjoy great popularity.

From 1962 he worked at the University of Sussex, of which he was one of the founders, and in 1973 he published his main contribution to game theory, known as the evolutionary stable strategy. His studies of the theory led to the publication of the book *Evolution and Theory of Games* (1982), which sets out what is known as the hawk-dove game. In 1977 he was made a member of the Royal Society and in 1986 received the Darwin medal, two of many recognitions of his work. The European Society for Evolutionary Biology created a prize in his honour for young researchers in the discipline.

the latter retreats (0). When two doves come into conflict, although there are no wounds, there is a great loss of time and unnecessary risk, and as such, Smith assigns a value of -3. In a conflict between doves, the winner gains 10 - 3 = 7 and the loser -3, making the average 2.

Based on this game, the idea of an evolutionary stable strategy is introduced, that is, one that remains despite any differing mutant behaviours that might arise. Using this strategy, Smith showed that both a population consisting solely of hawks and one consisting solely of doves, would be evolutionarily unstable. In line with the assigned payoffs, a mixed strategy with 8/13 hawks and 5/13 doves would give an evolutionary stable community, that is to say, would protect against any increases in both hawks and doves. It is easy to show that this is the case, however the difficulty lies in explaining how a group of can put this into practice. One solution is to imagine the existence of a 'hawk' gene in 8/13 of the population and another in the remainder, which causes individuals to act like doves, or even a single gene that causes one form of behaviour or another in these same proportions.

In the model described above, it is clear that neither of the two strategies is satisfactory: the hawks beat the doves but lose in confrontations between themselves, and the doves obtain a good performance when fighting among themselves but not when fighting against a hawk. A solution is required that reduces conflicts between hawks and at the same time impedes them from taking advantage of the fearful attitude of the doves, in other words maintaining their advantage over doves while reducing the number of violent confrontations between themselves. For this reason, this settlement is known as the 'bourgeois strategy'.

As an example of how the various applications support each other and provide inspiration for new ones, the idea of evolution was applied to game theory by Robert Axelrod in the study of cooperative strategies in a community where a given game is repeated a high number of times (see the box on page 127).

Games with more than two players

Until now, we have discussed two-player games, two people, two companies, two armies or, in general, any two groups. As such, the possibility of two or more players forming partnerships in order to improve their results at the cost of damaging the others was not feasible. The famous work by von Neumann and Morgenstern, *The*

Theory of Games and Economic Behavior was the first to deal with *n* person games and introduce ideas for their solution.

n person games

In order to introduce the main terminology from this part of von Neumann and Morgenstern's book on games for more than two players, and also to introduce their idea for finding a solution, we will use an example from the world of economics with highly simplified values. Three companies, E1, E2 and E3, each have a value of £1. Each can form a partnership with the others to make coalitions and the value of each coalition increases by £9. If two companies join forces, their value will be £11, whereas in the case of three, the value will be £12. Let us suppose that the three companies are equal in all respects. How should they form partnerships? Which coalition is preferable and how should the gains be distributed?

The above game is described in what is referred to as 'characteristic form'. Both the players and the coalitions have a stable value and when a coalition is formed it operates like a new player that replaces the two that have joined forces, making it possible to apply the methods for two-player games. Furthermore it is assumed that the coalition acts in order to maximise its gains and that if the game is a zero-sum game, this objective is achieved, as we have shown in previous chapters, by minimising those of the opponent. Let us also assume that once the coalitions have been formed, the game is wholly competitive.

We shall now analyse the results of this problem. If no partnerships are formed, each company remains in the initial condition, with a value of £1 each. If there is a partnership between the three companies (total value of £12), given the symmetry of the situation, an equal and satisfactory distribution for all participants is for each company to receive £4; this possibility is represented by the triplet (4, 4, 4), which gives the payoffs for each company and is referred to as the 'imputation'. However, other imputations are possible provided that the sum of the payments is £12. If the partnership is between two companies, e.g. B and C, the third (A) will now only receive £1 and the other two £11 in total; one possible imputation would be (1, 5.5, 5.5), although there are many others. Given that two companies increase their payoffs with respect to the previous imputation this seems more probable – since it is a better solution – than the first.

Although the solution (1, 5.5, 5.5) would appear to be the most feasible, it is unstable since company A, which has not managed to make a partnership, can

propose another, perhaps to B in which both obtain a greater payoff, for example (5,6,1). B can now try to intervene again, forcing smaller payoffs for A, with the same partnership, or C may propose a new one. This process could continue indefinitely and would struggle to reach a stable distribution in which the game could be considered to be solved.

Von Neumann and Morgenstern's analysis of *n* player games quickly led them to the conclusion that there was no single optimal solution, accepting that the solution was not determined by a single imputation. However, any analysis will show that not all imputations can form part of a solution, and this led them to try to define the conditions that must be met by the set of imputations that constitutes the solution to the game, the solution being understood as a set of imputations (payoffs for all players).

In order to understand the meaning of these conditions, it is necessary to make use of another concept, which is referred to as the 'dominance' of one imputation over another. If we understand that for each proposal for a coalition and its distribution, another arises, it follows that the new imputation of payoffs is not arbitrary, but is rationally better than the previous. This means that there must be a collection of players able to form a new coalition and an associated imputation of payoffs in which they receive a payoff that is strictly greater than the previous proposal.

Having defined the concepts of imputation and domination, it is now possible to formulate the conditions for determining the set of imputations that make up the solutions. There are essentially two:

1. All imputations that make up the solution may not be dominated by another which is also included in the solution.
2. All imputations that do not make up part of the solution must be dominated by one included in the solution.

Under these conditions, von Neumann and Morgenstern believed that the proposed solution, in addition to avoiding internal contradictions, was representative of socially acceptable behaviour. In order to be able to apply this method, there are a number of restrictions, the main one being that the players must be able to communicate with each other at all times, freely, in pairs as well as all together.

Cooperative games, partnerships and distributions

Continuing with *n* person games, let us now analyse a number of situations, progressively increasing the difficulty in each case, in which we assume that the players can communicate and make agreements prior to playing. As before, we wish to study which coalitions are possible and which ones guarantee a distribution of earnings such that all members of the coalition are satisfied and decide to continue with it.

Example 1

After closing a deal, three businesspeople, Anna (A), Beatrice (B) and Cedric (C), must distribute profits of £200,000 between themselves. They decide that the distribution will be by simple majority. Each will have one vote but there will be no other restriction on how the distribution is carried out. There are four possible coalitions that can achieve a simple majority: ABC, AB, AC, BC. However, each includes many different ways of distributing the profits between the three players.

Anna suggests the distribution A = £68,000, B = £66,000 and C = £66,000. Beatrice proposes another: A = £60,000, B = £70,000 and C = £70,000, which is better for her and Cedric. However Cedric proposes a third distribution: A = £70,000, B = 0 and C = £130,000, which increases the income for both himself and Anna. As in the previous section, the proposals could go on and on, and it would seem that there is no coalition able to satisfy the three partners: there is no point of equilibrium since each proposal could be modified by another to improve the payoffs received by each player in a new partnership.

In cooperative games with partnerships, a 'solution' is a proposal for a stable partnership and distribution of payoffs, that is to say, one that guarantees a satisfactory agreement between the members of the coalition.

Example 2

Let us now suppose that the decision regarding the previous distribution is taken in line with the investment made by each member such that Anna now has 5 votes, Beatrice has 3 and Cedric 1. Now the possible partnerships for achieving a majority are: ABC, AB, AC, A.

Since Anna has a majority, she can propose a distribution in which she obtains all the profit: A = £200,000, B = 0 and C = 0. Although the distribution is not fair, it is stable. Anna will be in favour and it is impossible to form a partnership without

her; as such, this is a solution which meets the definition we have just provided.

In this type of game, the value of the game is the payoff guaranteed to each player if they act in a rational manner and is independent of the decisions of the other players. In example 1, none of the players is guaranteed to receive anything, meaning that the value of the game is A = 0, B = 0 and C = 0. On the other hand, in the second example, the value of the game is A = 100, B = 0 and C = 0.

Example 3

Let us now complicate matters even further in order to bring the situation closer to the real world. In an election, 5 political parties have 81 seats between them, which are distributed in the following manner: A = 33, B = 24, C = 15, D = 6, E = 3. Given that none of the parties has an absolute majority (41 seats), a partnership or coalition is required to form a government. The coalition will decide the distribution of the budget and the allocation of responsibilities. Ideological affinities are disregarded and it is assumed that the importance of the positions depends on the budget for which they are responsible. Finally, voting discipline is guaranteed.

Of all the possible political partnerships (1 with 5 parties, 5 with 4, 10 with 3, 10 with 2 and 5 with just one) there are 16 that are feasible (have a minimum of 41 seats). As none of the political parties has a majority, the value of the game for

> **LLOYD STOWELL SHAPLEY (1923–)**
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> Shapley was an important mathematician and economist from the United States who made a number of fundamental contributions to game theory. He studied mathematics at Harvard, graduating in 1948 after serving in the Second World War as an army air corps sergeant stationed in China. He spent a year at the RAND Corporation and received his PhD from Princeton University in 1953 at a time when some of the most eminent game theorists were working there. He then returned to work at the RAND Corporation until 1981, when he joined the teaching staff of the University of California, Los Angeles (UCLA).
>
> In his PhD thesis, he was already beginning to introduce certain concepts such as the Shapley value, and throughout the course of his extended career, he continued to publish results from his initial research. He has been a member of the National Academy of Sciences since 1979 and has received a number of prizes, including the John von Neumann Theory Prize in 1981.

each party is 0, since no party is essential in the formation of a coalition with the ability to govern.

In situations such as the one described above, the mathematician and economist Lloyd Shapley proposed a distribution system, the values of which were proportional to the number of possible winning partnerships in which the player's participation is decisive (without it, the partnership would no longer be able to win). The payment received by each player is referred to as the Shapley value. A player is not decisive in a coalition when they are not essential to ensuring its success.

In our example, in a coalition with all the parties, none is decisive, whereas in a coalition between BCDE, B and C are decisive, since if either withdraw from the coalition, the remainder no longer has a majority (if B withdraws, the coalition will be left with just 24 seats and if C withdraws it will have 33). On the other hand, D and E are not decisive since if one of these withdraws from the coalition, it retains its majority (if D withdraws, the coalition will have 42 seats and if E withdraws it will have 45). Under these conditions, and counting accordingly, the number of coalitions in which each party is decisive can be summarised by the following table:

Party	Number of coalitions in which the party is decisive
A	10
B	6
C	6
D	2
E	2

Under these conditions, it is possible to carry out the distribution in line with the model proposed by Shapley. If a partnership is formed between all the parties and the budget is £2,600 million, the distribution based on the Shapley value would be (in millions):

A = 1,000
B = 600
C = 600
D = 200
E = 200.

In any other partnership, each of the participating parties will receive a budget distributed in the same way. However it will never be less than the value obtained in this coalition. This proposal for the distribution does not give the only stable solution, since there are also a number of other possibilities. However, at any rate, in any coalition that is formed, if the distribution is carried out in this manner, there will be no other stable possibility that offers the participants a higher value.

Both the method proposed by von Neumann and the one proposed by Shapley show, on the one hand, that the solution is not expressed by a single imputation, but by a set of them, and on the other, that it is possible to give a set of properties that make it possible to decide if a given imputation forms part of the 'solution' set.

Throughout the course of these last two chapters, the reader will have observed that as the situations analysed grow more complex, becoming increasingly close to real-world scenarios, the mathematical methods applied in an attempt to solve the problems become less convincing. This does not mean that they are no longer equally as valid, just that real world situations that combine elements of conflict and cooperation each have certain specific properties. This means that the mathematical methods that can be applied in an attempt to solve them must account for the fact that their validity depends on these properties.

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Choices and Strategies

Knowing how to play the game

Game playing – an ever-popular pastime – has always prompted intriguing mathematical explanations and formulations. This process reached its climax in the second half of the last century when, in the heat of the Cold War, against a backdrop of ideological and military confrontation between global superpowers, modern game theory was developed to analyse winning strategies that can be used to tackle all types of conflicts.